Contributions to Infinite Divisibility for Financial Modeling

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Contributions to Infinite Divisibility for Financial Modeling

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To my parents

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Summary

Infinitely divisible distributions and processes have been the object of extensive research not only from the theoretical point of view but also for practical use, for example, in queueing theory or mathematical finance. In this thesis, we will study some of their subclasses with a view towards financial modeling. As generalizations of stable distributions, we study the tempered stable distributions of Rosiński [41], and introduce the new classes of layered stable distributions as well as the mixed stable distributions, along with the corresponding Lévy processes. As a further generalization of infinitely divisible processes, fractional tempered stable motions are defined. These theoretical studies will be complemented by some more practical ones, such as the simulation of sample paths, parameter estimations, financial portfolio hedging, and solving stochastic differential equations.

The first chapter of this thesis is devoted to reviewing some basic facts on infinitely divisible distributions and processes. In particular, series representations and some elements of the Malliavin calculus are presented.

Chapter 2 reviews and further investigates tempered stable distributions and tempered stable Lévy processes. Their asymptotic short-time stable-like behavior and long-time Gaussian-like behavior as well as their probability tail behaviors are studied. Moreover, their series representations are derived by the generalized shot noise method and the rejection method. Their fits to real stock prices are discussed.

Chapter 3 introduces fractional tempered stable motions. They are defined as a stochastic integral of a reproducing Volterra kernel with respect to tempered stable

Lévy motions. It is shown that they possess the same covariance structure as fractional Brownian motions and are second-order selfsimilar infinitely divisible processes with second-order stationary increments and whose marginal laws are tempered stable. Their asymptotic short-time and long-time behaviors, which are inherited from background driving tempered stable Lévy processes, and non-semimartingale sample path property are proved. Parameter estimation results are presented with the help of sample path generations via series representations.

Chapter 4 introduces two generalizations of stable distributions and their corresponding Lévy processes; layered stable processes and mixed stable processes. Their basic properties, such as moments and series representations, are obtained. In addition, the short-time and long-time behaviors of layered stable processes are proved. Furthermore, their tempered versions are also studied.

Chapter 5 is an empirical study. A variance reduction method in Monte Carlo simulation and a numerical method for solving stochastic differential equations driven by pure-jump additive processes are proposed. Moreover, a stock price model based on additive processes is introduced. The applicability of the model is discussed in terms of minimal variance hedging with the help of Malliavin calculus techniques. The numerical results indicate a fairly reliable accuracy in hedging replications, when compared with a Lévy process model and the classical Black-Scholes model.

Chapter 1

General Preliminaries

1.1 Notation

Throughout this thesis, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ will be a filtered probability space satisfying the usual conditions, i.e.,

(1) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

(2) the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous.

Moreover, we assume that $\mathcal{F} = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$, i.e., $\mathcal{F}$ is the smallest σ-field containing all the $\mathcal{F}_t$'s. This allows us to specify a change of the underlying probability measure P to a probability measure Q by giving a density process $\{Z_t : t \geq 0\}$ where $dQ|_{\mathcal{F}_t} := Z_t dP|_{\mathcal{F}_t}$.

Let $\{X_t : t \geq 0\}$ be a stochastic process in $\mathbb{R}^d$. X is said to be *càdlàg* (continu à droite, limites à gauche in French) if, almost surely, it has sample paths which are right-continuous with left limits. We will denote by $\mathbb{D}$ the space of càdlàg functions. On the other hand, if it almost surely has left-continuous sample paths with right limits, then it is said to be *càglàd*. We say that X is *stochastically continuous*, or *continuous in probability*, if for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s\to t} P(\|X_s - X_t\| > \epsilon) = 0.$$

We denote by ΔX_t the jump size of X at time t, by $[X,X]_t$ the *quadratic variation*

$$[X,X]_t = X_t^2 - 2\int_0^t X_{s-}dX_s$$

and by $[X,X]_t^c$ the path-by-path continuous part of $[X,X]_t$, i.e.,

$$[X,X]_t^c = [X,X]_t - X_0^2 - \sum_{0<s\le t}(\Delta X_s)^2.$$

Finally, $\stackrel{d}{=}$ indicates the equality in law. We will also write $\{X_t : t \ge 0\} \stackrel{d}{=} \{Y_t : t \ge 0\}$ when they have same finite dimensional distributions, and $\{X_t^n : t \ge 0\}_{n\ge 1} \xrightarrow{d} \{Y_t : t \ge 0\}$ when finite dimensional distributions of X^n converges to those of Y as $n \to \infty$.

1.2 Infinite Divisibility

Let us begin with the definition of infinitely divisible distributions. Here, μ^{*n} denotes the n-fold convolution of probability measure μ with itself.

Definition 1.2.1. *A probability measure μ on $\mathbb{R}^d$ is said to be infinitely divisible if, for each $n \in \mathbb{N}$, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{n*}$.*

Every infinitely divisible distribution has a unique representation in the form of characteristic function. Let μ be an infinitely divisible distribution on $\mathbb{R}^d$. Then, its characteristic function $\widehat{\mu}$ is given by

$$\widehat{\mu}(y) = \exp\left[i\langle y,\gamma\rangle - \frac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle 1_{\{\|z\|\le 1\}}(z))\nu(dz)\right], \quad (1.2.1)$$

where $\gamma \in \mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix and ν is a measure on $\mathbb{R}_0^d$ such that

$$\int_{\mathbb{R}_0^d}(\|z\|^2 \wedge 1)\nu(dz) < \infty, \quad (1.2.2)$$

and $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$. In addition, the representation (1.2.1) by γ, A and ν is unique. Conversely, if γ, A and ν are given as above, then they uniquely construct a infinitely divisible distribution with the characteristic function (1.2.1). This result is called the *Lévy-Khintchine representation* of infinitely divisible distributions. We will call (γ, A, ν) the *generating triplet* of the infinitely divisible distribution μ. The A and the ν are called the *Gaussian covariance matrix* and ν the *Lévy measure*, respectively. If $A = 0$, then μ is said to be purely non-Gaussian.

Let us state an important result on the finiteness of the moments of infinitely divisible distributions that we are going to use frequently in the sequel. Here, a function $g(x)$ on $\mathbb{R}^d$ is said to be *submultiplicative* if it is nonnegative and there is a constant $a > 0$ such that $g(x + y) \leq ag(x)g(y)$ for $x, y \in \mathbb{R}^d$.

Theorem 1.2.2. (Kruglov [24]) *Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ with Lévy measure ν. Then, for a locally bounded submultiplicative function g on $\mathbb{R}^d$, $\int_{\mathbb{R}^d} g(x)\mu(dx) < \infty$ if and only if $\int_{\|z\|>1} g(z)\nu(dz) < \infty$.*

Here, $(\|x\| \vee 1)^p$ with $p > 0$ and $e^{\|x\|}$ are important examples of locally bounded submultiplicative functions. In particular, the first example induces the important equivalence; $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ if and only if $\int_{\|z\|>1} \|z\|^p \nu(dz) < \infty$.

We close this section by giving the definition of *infinitely divisible processes*. All stochastic processes that we are going to discuss in this thesis are in this class.

Definition 1.2.3. *A stochastic process is said to be infinitely divisible if all its finite-dimensional distributions are infinitely divisible.*

1.3 Lévy Processes and Additive Processes

Lévy processes and additive processes are most important examples of infinite divisible processes. Let us first give their definition.

Definition 1.3.1. *A stochastic process is called an additive process if all the following conditions are satisfied:*

(i) it has independent increments,

(ii) $X_0 = 0$ *a.s.,*

(iii) it is stochastically continuous,

(iv) its sample path is almost surely right-continuous with left limits.

Moreover, an additive process with independent increments is called an Lévy process.

It is not immediate from the definition that they are infinitely divisible processes. But, it can be seen as follows. By the *Lévy-Itô decomposition*, every additive process $\{X_t : t \geq 0\}$ in $\mathbb{R}^d$ admits the a.s. canonical integral representation

$$X_t = b_t + G_t + \int_0^t \int_{\|z\| \leq 1} z(\varsigma - \varrho)(dz, ds) + \int_0^t \int_{\|z\|>1} z\varsigma(dz, ds), \tag{1.3.1}$$

where b is some continuous function with $b_0 = 0$, $\{G_t : t \geq 0\}$ is a centered Gaussian process in $\mathbb{R}^d$ with independent increments, Gaussian covariance matrix $\{H_t : t \geq 0\}$ and $G_0 = 0$ and ς is a Poisson random measure on $\mathbb{R}_0^d \times (0, \infty)$, independent of G, with intensity measure ϱ satisfying, for each $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}_0^d} (\|z\|^2 \wedge 1)\varrho(dz, ds) < \infty.$$

Then, for every $t \geq 0$, the characteristic function of X_t is given by

$$\widehat{\mu}_{X_t}(y) = \exp\left[i\langle y, b_t\rangle - \frac{1}{2}\langle y, H_t y\rangle + \int_0^t \int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle 1_{\{\|z\|\leq 1\}}(z))\varrho(dz, ds) \right],$$

which is clearly a characteristic function of an infinitely divisible distribution. Moreover, notice that X becomes a Lévy process if one sets $b_t = \gamma t$ for some $\gamma \in \mathbb{R}^d$, $H_t = tA$ for some Gaussian covariance matrix A, and $\varrho(dz, ds) = \nu(dz)ds$ for some Lévy measure ν. Then, we get

$$\widehat{\mu}_{X_t}(y) = \exp\left[t\left(i\langle y, \gamma\rangle - \frac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle 1_{\{\|z\|\leq 1\}}(z))\nu(dz) \right)\right], \tag{1.3.2}$$

and

$$X_t = \gamma t + B_t + \int_0^t \int_{\|z\| \leq 1} z(\mu - \nu)(dz, ds) + \int_0^t \int_{\|z\|>1} z\mu(dz, ds), \tag{1.3.3}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion with covariance matrix A and μ is a Poisson random measure with intensity measure ν. The uniqueness of the characteristic functions and the independence of increments imply that the law $\mathcal{L}(X)$ of a Lévy process X is determined by $\mathcal{L}(X_1)$. In fact, we have $E[e^{i\langle y, X_t\rangle}] = E[e^{it\langle y, X_1\rangle}]$. As seen so far, the additive process can be considered to be a simple generalization of the Lévy process. Thus, for notational convenience, we will henceforth put more attention to Lévy processes.

Let us close this section by summarizing some basic facts. If $0 < \nu(\mathbb{R}_0^d) < \infty$, then the total number of all jumps up to time 1, i.e. $\mu(\mathbb{R}_0^d \times [0,1])$, is simply a Poisson random variable with parameter $\nu(\mathbb{R}_0^d)$. However, if $\nu(\mathbb{R}_0^d) = \infty$, then the jumping times are only countable and dense in the time interval $[0, \infty)$ a.s., and the sum of all jumps up to time t may diverge. The compensation in the third term of (1.3.3) resolves this problem. If $\int_{\|z\|\leq 1} \|z\|\nu(dz) < \infty$, then the sum of all jumps converges and thus (1.3.2) and (1.3.3) can be rewritten as, respectively,

$$E[e^{i\langle y, X_t\rangle}] = \exp\left[t\left(i\langle y, \beta\rangle - \frac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y, z\rangle} - 1)\nu(dz)\right)\right],$$

$$X_t = \beta t + B_t + \int_0^t \int_{\mathbb{R}_0^d} z\mu(dz, ds),$$

where $\beta = \gamma - \int_{\|z\|\leq 1} z\nu(dz)$. On the other hand, when $\int_{\|z\|>1} \|z\|\nu(dz) < \infty$, the representations can be written as

$$E[e^{i\langle y, X_t\rangle}] = \exp\left[t\left(i\langle y, \beta_1\rangle - \frac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y, z\rangle} - 1 - i\langle y, z\rangle)\nu(dz)\right)\right], \quad (1.3.4)$$

$$X_t = \beta_1 t + B_t + \int_0^t \int_{\mathbb{R}_0^d} z(\mu - \nu)(dz, ds),$$

where $\beta_1 = \gamma + \int_{\|z\|>1} z\nu(dz)$. When (1.3.4) holds, we will say that $\{X_t : t \geq 0\}$ is a Lévy process generated by $(\beta_1, A, \nu)_1$. In this case, the pure jump component is compensated and thus is a martingale. Let us call a Lévy process generated by $(0, A, \nu)_1$ a *Lévy martingale.*

1.4 Series Representations

The series representations of purely non-Gaussian infinitely divisible distributions is established by Khintchine [21] with, so called, the inverse Lévy measure method, which is a special case of the LePage's method [25]. Ferguson and Klass [13] rediscovered the inverse Lévy measure method and applied it to stochastic processes with independent increments. Since then, several generalizations have been proposed. Just recently, Rosiński [40] surveyed those methods and properties for Lévy processes. In this thesis, we will extensively use the concept of series representations for a simulation of sample paths and studies of their structure. For the sake of completeness, let us summarize general theorems of the series representation with a brief sketch of the proof.

Theorem 1.4.1. (Rosiński [40]) *Let ν be a Lévy measure on $\mathbb{R}_0^d$ admitting the disintegration form*

$$\nu(B) = \int_0^\infty P(H(r,V) \in \cdot)dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{1.4.1}$$

where V is a random variable in a measurable space S with distribution F, and H is a function from $(0,\infty) \times S$ to $\mathbb{R}_0^d$ such that for each $v \in S$, $r \to |H(r,v)|$ is nondecreasing. Let μ be a Poisson random measure on $\mathbb{R}_0^d \times (0,\infty)$ with intensity measure ν. Then, we have the following.

(i) The following equality holds for finite dimensional distributions;

$$\left\{\int_0^t \int_{\|z\|\le 1} z(\mu-\nu)(dz,ds) + \int_0^t \int_{\|z\|>1} z\mu(dz,ds) : t \in [0,T]\right\}$$
$$\stackrel{d}{=} \left\{\sum_{i=1}^\infty \left[H(\Gamma_i/T, V_i)1(T_i \le t) - c_i\frac{t}{T}\right] : t \in [0,T]\right\},$$

where the convergence of the right hand side is a.s. uniformly on $[0,T]$. Here, $\{\Gamma_i\}_{i\ge1}$ are arrival times of a standard Poisson process, $\{V_i\}_{i\ge1}$ is a sequence of iid random variables in S with common distribution F, $\{T_i\}_{i\ge1}$ are iid uniform in $[0,T]$, and

$\{c_i\}_{i\geq 1}$ is a sequence of constants given by

$$c_i = \int_{i-1}^{i} E[H(s/T, V_1)1(\|H(s/T, V_1)\| \leq 1)]ds. \tag{1.4.2}$$

(ii) If $\int_{\|z\|\leq 1} \|z\|\nu(dz) < \infty$, then

$$\left\{ \int_0^t \int_{\mathbb{R}_0^d} z\mu(dz, ds) : t \in [0, T] \right\} \stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} H(\Gamma_i/T, V_i)1(T_i \leq t) : t \in [0, T] \right\}.$$

(iii) If $\int_{\|z\|>1} \|z\|\nu(dz) < \infty$, then

$$\left\{ \int_0^t \int_{\mathbb{R}_0^d} z(\mu - \nu)(dz, ds) : t \in [0, T] \right\}$$
$$\stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[H(\Gamma_i/T, V_i)1(T_i \leq t) - d_i \frac{t}{T} \right] : t \in [0, T] \right\},$$

where $\{d_i\}_{i\geq 1}$ is a sequence of constants given by

$$d_i = \int_{i-1}^{i} E[H(s/T, V_1)]ds. \tag{1.4.3}$$

(iv) There exist random sequences $\{\widetilde{\Gamma}_i\}_{i\geq 1}$, $\{\widetilde{V}_i\}_{i\geq 1}$, and $\{\widetilde{T}_i\}_{i\geq 1}$ defined on the same probability space as μ satisfying $\{\widetilde{\Gamma}_i\}_{i\geq 1} \stackrel{d}{=} \{\Gamma_i\}_{i\geq 1}$, $\{\widetilde{V}_i\}_{i\geq 1} \stackrel{d}{=} \{V_i\}_{i\geq 1}$, $\{\widetilde{T}_i\}_{i\geq 1} \stackrel{d}{=} \{T_i\}_{i\geq 1}$, and

$$\int_0^t \int_{\|z\|\leq 1} z(\mu - \nu)(dz, ds) + \int_0^t \int_{\|z\|>1} z\mu(dz, ds)$$
$$= \sum_{i=1}^{\infty} \left[H(\widetilde{\Gamma}_i/T, \widetilde{V}_i)1(\widetilde{T}_i \leq t) - c_i \frac{t}{T} \right] \quad a.s.$$

The same also holds for the cases (ii) and (iii).

Sketch of proof. Let us first consider the case $T = 1$. Define a marked Poisson process $\widetilde{\mu}$ on $(0, \infty) \times S \times [0, 1]$ by

$$\widetilde{\mu} := \sum_{i=1}^{\infty} \delta_{(\Gamma_i, V_i, T_i)},$$

whose intensity measure is given by $dr\, F(dv)ds =: \widetilde{\nu}(dr, dv)ds$. Observing that $\widetilde{\nu} \circ H^{-1} = \nu$, we have $\widetilde{\mu}(H^{-1}(B), C) \stackrel{d}{=} \mu(B, C)$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$ and $C \in \mathcal{B}([0, 1])$. But, we can also define μ on the same probability space as $\widetilde{\mu}$, by

$$\mu(B,C) := \sum_{i=1}^{\infty} \delta_{(H(\Gamma_i,V_i),T_i)}(B,C),$$

which implies that $\widetilde{\mu}(H^{-1}(B),C) = \mu(B,C)$ a.s. Then, we can transform the Lévy-Itô decomposition to a Poisson integral representation with respect to $\widetilde{\mu}$ as

$$\begin{aligned} X_t &:= \int_0^t \int_{\|z\|\le 1} z(\mu-\nu)(dz,ds) + \int_0^t \int_{\|z\|>1} z\mu(dz,ds) \\ &= \int_0^t \int_S \int_0^\infty H(r,v)1(\|H(r,v)\| \le 1)(\widetilde{\mu}-\widetilde{\nu})(dr,dv,ds) \\ &\qquad + \int_0^t \int_S \int_0^\infty H(r,v)1(\|H(r,v)\| > 1)\widetilde{\mu}(dr,dv,ds). \end{aligned}$$

Set a stochastic process

$$\begin{aligned} X_t^n &:= \int_0^t \int_S \int_0^n H(r,v)1(\|H(r,v)\| \le 1)(\widetilde{\mu}-\widetilde{\nu})(dr,dv,ds) \\ &\qquad + \int_0^t \int_S \int_0^n H(r,v)1(\|H(r,v)\| > 1)\widetilde{\mu}(dr,dv,ds). \end{aligned}$$

Since the number of arrivals of a standard Poisson process on every compact set is finite a.s., we get

$$\begin{aligned} X_t^n &= \sum_{\{i:\Gamma_i\le n\}} H(\Gamma_i,V_i)1(T_i\le t) - t\int_0^n \int_S H(r,v)1(\|H(r,v)\|\le 1)\widetilde{\nu}(dr,dv) \\ &= \sum_{i=1}^{m(n)} \left[H(\Gamma_i,V_i)1(T_i\le t) - t\int_{i-1}^i \int_S H(r,v)1(\|H(r,v)\|\le 1)\widetilde{\nu}(dr,dv) \right] \\ &\qquad + t\int_n^{m(n)} \int_S H(r,v)1(\|H(r,v)\|\le 1)\widetilde{\nu}(dr,dv), \end{aligned}$$

where $m(n) := \max\{i \in \mathbb{N} : \Gamma_i \le n\}$. Clearly, $\{X_t^n : t \in [0,1]\}$ has a càdlàg path and $m(n) \to \infty$ a.s. as $n \to \infty$. Moreover, the construction of $\{X_t^n : t \in [0,1]\}$ implies that $X_t^n \to X_t$ a.s. The second term of the right hand side can be proved to converge to 0 as $n \to \infty$. Moreover, the results of Kallenberg [19] imply that $\lim_{k\to\infty} \sup_{k\le i<l} \|X^l - X^k\|_{\mathbb{D}} = 0$ a.s., where $\|\cdot\|_{\mathbb{D}}$ denotes the supremum norm of a càdlàg function on $[0,1]$. This gives the almost sure convergence of the infinite

sum uniformly on $[0,1]$. (See Theorem 4.1(B) and 5.1 of Rosiński [40].) Finally, the result can be easily extended to arbitrary $T > 0$ by observing that $\nu(B)Leb(C) = \int_0^\infty P(H(r/T,V) \in B)dr\, T^{-1}Leb(C)$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$ and $C \in \mathcal{B}([0,T])$.

The method based on this is called the *generalized shot noise method* [40]. Indeed, several known methods are its special cases. Let us state two such methods that we are going to use in this thesis. Here, when a Lévy process $\{X_t : t \in [0,T]\}$ with Lévy measure ν admits a series representation $\sum_{i=1}^\infty [H_i 1(T_i \le t) - c_i t/T]$, we will call the random sequence $\{H_i\}_{i\ge1}$ a *H-sequence* of ν.

(i) LePage's method [25]; Suppose that a Lévy measure ν on $\mathbb{R}_0^d$ admits the following disintegration form in polar coordinates;

$$\nu(B) = \int_{S^{d-1}} \lambda(\xi) \int_0^\infty 1_B(r\xi) q(dr,\xi), \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where λ is a probability measure on S^{d-1} and for each $\xi \in S^{d-1}$, $q(\cdot,\xi)$ is a Lévy measure on $(0,\infty)$. Define

$$\overleftarrow{q}(u,\xi) := \inf\{r > 0 : q([r,\infty),\xi) < u\}.$$

Then, a H-sequence is given by

$$H_i = \overleftarrow{q}(\Gamma_i/T, V_i)V_i, \quad i \ge 1,$$

where $\{V_i\}_{i\ge1}$ is a sequence of iid random vectors in S^{d-1} with common distribution λ. Let us call the $\overleftarrow{q}$ the *inverse q-function.*

(ii) Rejection Method [40]; Let ν_0 be a Lévy measure on $\mathbb{R}_0^d$ such that

$$\frac{d\nu}{d\nu_0} \le 1,$$

with $\{H_i^0\}_{i\ge1}$ as its H-sequence. Then, an H-sequence $\{H_i\}_{i\ge1}$ of ν is given by

$$H_i = \begin{cases} H_i^0, & \text{if } \frac{d\nu}{d\nu_0}(H_i^0) \ge U_i \\ 0, & \text{otherwise,} \end{cases}$$

where $\{U_i\}_{i\geq 1}$ is a sequence of iid uniform in $[0,1]$.

Clearly, for every infinitely divisible random vector ξ in $\mathbb{R}^d$, there is a Lévy process $\{X_t : t \geq 0\}$ in $\mathbb{R}^d$ such that $\xi \stackrel{d}{=} X_1$. Hence, by setting $T = t = 1$ above, we get a series representations of purely non-Gaussian infinitely divisible distributions. This, however, does not give a good method for the simulation unless the series converges extremely fast; In order to generate one random vector, the series representation method requires us to sum many random vectors and variables.

They can also be extended to additive processes. Define an additive process $\{X_t : t \in [0,T]\}$ by

$$X_t := \int_0^t \int_{\|z\|\leq 1} z(\varsigma - \varrho)(dz, ds) + \int_0^t \int_{\|z\|>1} z\varsigma(dz, ds),$$

where ς is a Poisson random measure on $\mathbb{R}_0^d \times [0,T]$ with intensity measure ϱ. Here, we assume that ϱ can be decomposed as $\varrho(B,C) := \nu(B)K(C)$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$ and $C \in \mathcal{B}([0,T])$, where $K(C) := \int_C y(s)ds$ with $y : [0,T] \to (0,\infty)$. We will call the y the *timer*. Let $\{S_i\}_{i\geq 1}$ be a sequence of iid random variables in $[0,T]$ with common distribution $G(C) := K(C)/K([0,T])$ for $C \in \mathcal{B}([0,T])$.

Proposition 1.4.2. $\{X_t : t \in [0,T]\}$ *admits a series representation*

$$X_t = \sum_{i=1}^{\infty} \Big[H(\Gamma_i/K([0,T]), V_i)1(S_i \leq t) - c_i(K([0,T]))G([0,t]) \Big] \quad a.s.$$

Proof. A simple modification of the proof of Theorem 1.4.1 will give the result. For each $t \in [0,T]$, the equality in law and the a.s. convergence of the sum readily follow because X_t is simply a well defined infinitely divisible random variable. Observe that for $B \in \mathcal{B}(\mathbb{R}_0^d)$ and $C \in \mathcal{B}([0,T])$,

$$\begin{aligned} \varrho(B,C) &= \int_0^\infty P(H(r,V) \in B) dr\, K(C) \\ &= \int_0^\infty P(H(r/K([0,T]),V) \in B) dr\, G(C). \end{aligned}$$

This implies that we only need to replace a sequence of iid uniform in $[0,T]$ by that of random variables whose distribution is G, and also replace Γ_i/T by $\Gamma_i/K([0,T])$. The a.s. uniform convergence on $[0,T]$ and the a.s. equality follow as in the homogeneous case.

Example 1.4.3. In Figure 1.1, we give typical sample paths of two additive processes. In the left figures, the timer is set as

$$y(t) := \begin{cases} 1, & if\ t \in [0,1] \cup (2,3] \\ 5, & if\ t \in (1,2]. \end{cases}$$

In the right figures, $y(t) := t$, $t \in [0,1]$. Their original Lévy process $\{X_t : t \in [0,1]\}$ is set to be the one dimensional symmetric CGMY process with parameters $(C,G,M,Y) = (0.01, 1.0, 1.0, 1.97)$. (We shall define CGMY processes later.) Its Lévy measure ν is given by

$$\nu(dz) = C\left(\frac{e^{-G|z|}}{|z|^{Y+1}}1(z<0) + \frac{e^{-Mz}}{z^{Y+1}}1(z>0)\right)dz, \quad z \in \mathbb{R}_0.$$

The generalized shot noise method yields the following series representation; for $t \in [0,T]$,

$$X_t = \sum_{i=1}^{\infty}\left(m(Y\Gamma_i/T)^{-1/Y} \wedge E_i U_i^{1/Y}|V_i|\right)\frac{V_i}{|V_i|}1(S_i \le t) \quad a.s.$$

where $m = (C(1/G^Y + 1/M^Y))^{1/Y}$, $\{E_i\}_{i\ge1}$ are iid exponential with parameter 1, $\{U_i\}_{i\ge1}$ are iid uniform in $[0,1]$, $\{V_i\}_{i\ge1}$ is a sequence of iid random variables with distribution

$$\rho_1(dx) := C/m^Y(1/G^Y\delta_{-1/G}(dx) + 1/M^Y\delta_{1/M}(dx)),$$

and $\{S_i\}_{i\ge1}$ are iid random variables in $[0,T]$ induced by the timers. On simulating the law of $\{X_t\}$, we need to truncate the summation by some finite number, i.e., discarding jumps with smaller absolute size. In this example, the summation is taken up to $N = 4000$ (i.e., $\sum_{i=1}^{4000}$). Both noise processes do give an good intuition of the influence of the timers. It can be observed that the noise process in the left has high variations on $(1, 2]$ while the magnitude of the fluctuation grows in time on the right side.

Remark 1.4.4. Timers can be stochastic as long as independent of the random sequences $\{\Gamma_i\}_{i\geq 1}$ and $\{V_i\}_{i\geq 1}$. In such a case, we have only to generate the stochastic timer $\{y_t : t \in [0, T]\}$ in advance and then obtain the distribution G on $[0, T]$ accordingly.

The series representations directly construct a sample path, i.e. the jump size at time S_i is given by $H(\Gamma_i/T, V_i)$. It is thus natural to expect that the simulation by series representations capture the dynamics of sample paths more precisely than the Euler scheme. This fact is more apparent for additive processes. For example, set a timer $y(t) := t$ for $t \in [0, T]$. Then, ϱ is given by $\varrho(B, [0, t]) = \nu(B) \int_0^t y(s)ds = \nu(B)t^2/2$, $B \in \mathcal{B}(\mathbb{R}_0^d)$. In the Euler scheme, one approximates the timer by a step function (e.g., $y(t) = n/N$, $t \in [(n-1)/N, n/N)$ in this case), and generates a random variable for each interval. In contrast to the Lévy process case, however, random variables for two different intervals are independent but, in general, *not* identically distributed. Thus, one is required to compute many different probability distributions. On the other hand, one tries to enhance the accuracy by approximating timers with finer mesh. But, this is another pitfall. Indeed, as each interval gets shorter, the Fourier inversion gives very distorted density functions.

In simulation on the computer, we need to somehow truncate the infinite sum. In view of the fact that $\|H(r, \cdot)\|$ is nonincreasing in r, the truncation essentially corresponds to discarding jumps with smaller absolute size. Asmussen and Rosiński [2] derived a necessary and sufficient condition under which discarded small jumps in total are asymptotically Gaussian in the one dimension. Here, we extend it to the multidimensional setting. Let $\{X_t : t \geq 0\}$ be a Lévy process in $\mathbb{R}^d$ generated by

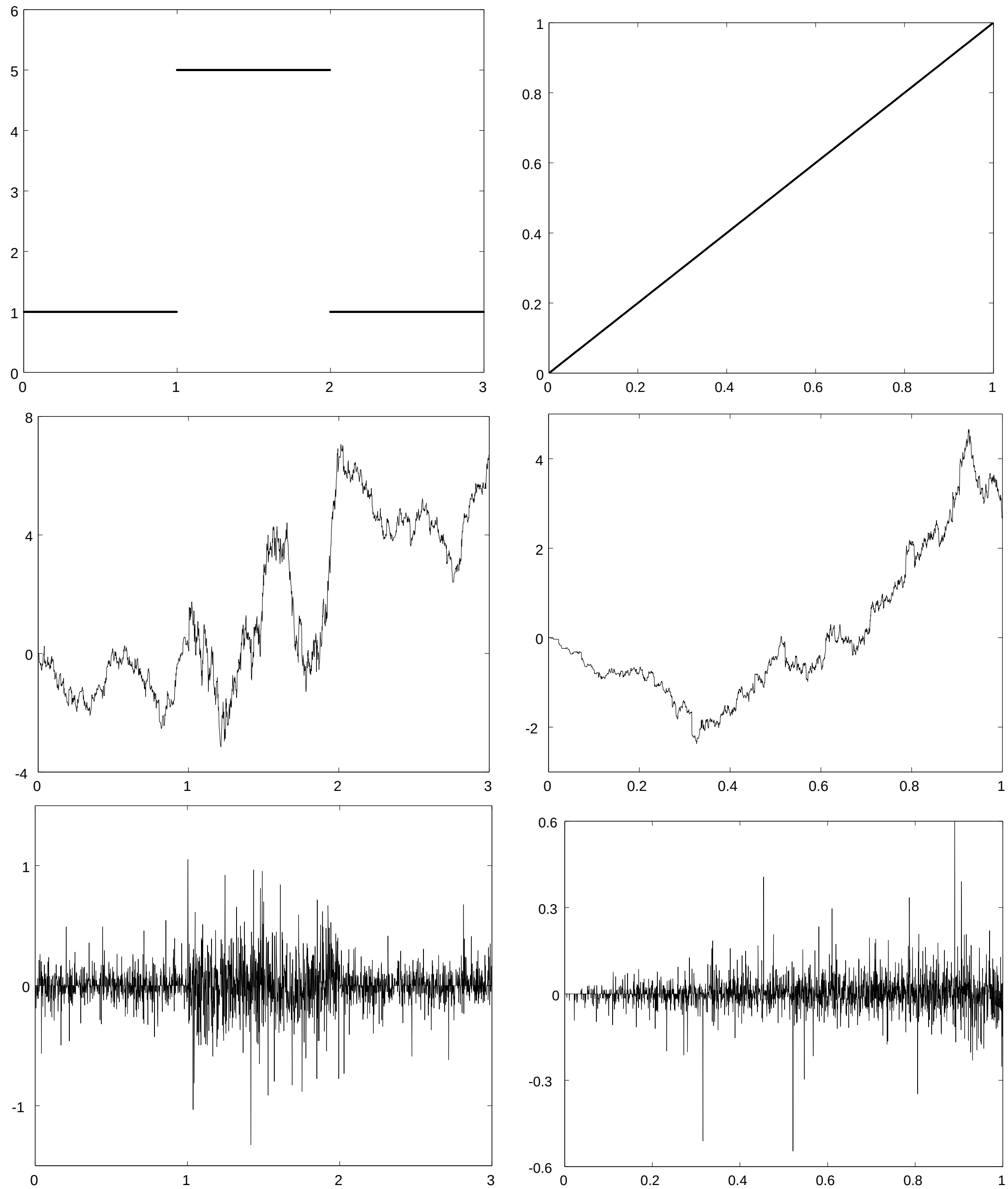


Figure 1.1: Timers (top), typical sample paths of additive process (middle), and their noise processes (bottom)

$(0, 0, \nu)$ and $\{X_t^\epsilon : t \geq 0\}$ a discarded small jump component whose characteristic function is given by

$$E[e^{i\langle y, X_t^\epsilon\rangle}] = \exp\left[t \int_{\|z\|\leq\epsilon} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle)\nu(dz)\right].$$

Set also $A_\epsilon = \int_{\|z\|<\epsilon} zz^T\nu(dz)$, where the superscript T denotes the matrix transposition. Since A_ϵ is nonnegative-definite for every $\epsilon > 0$, there exists a unique lower triangular matrix $M_\epsilon \in \mathbb{R}^{d\times d}$ with positive diagonal entries such that $A_\epsilon = M_\epsilon M_\epsilon^T$. Here, $\|\cdot\|_O$ denotes an operator norm.

Proposition 1.4.5. *If $\|\epsilon^2 A_\epsilon^{-1}\|_O \to 0$ as $\epsilon \to 0$, then $\{X_t^\epsilon : t \geq 0\}$ is approximately a Brownian motion with covariance matrix A_ϵ as $\epsilon \to 0$.*

Proof. Clearly, $\|\epsilon^2 A_\epsilon^{-1}\|_O \to 0$ is equivalent to $\|\epsilon M_\epsilon^{-1}\|_O \to 0$. Set $Y^\epsilon := (M_\epsilon^T)^{-1}X^\epsilon$ and $S_\epsilon := \{z \in \mathbb{R}_0^d : \|z\| \leq \epsilon\}$. Then, $\{Y_t^\epsilon : t \geq 0\}$ is a Lévy process with characteristic function

$$E[e^{i\langle y,Y_t^\epsilon\rangle}] = \exp\left[t\left(i\langle y, b_\epsilon\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle 1_{\{\|z\|\leq 1\}}(z))\nu_\epsilon(dz)\right)\right],$$

where $\nu_\epsilon(B) := \nu(M_\epsilon^T B \cap S_\epsilon)$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$, and $b_\epsilon := -(M_\epsilon^T)^{-1}\int_{\|z\|>1} z\nu_\epsilon(dz)$. It suffices to show that $Y_1^\epsilon \xrightarrow{d} B_1$ as $\epsilon \to 0$ where $\{B_t : t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}^d$. Moreover, Theorem 15.4 of Kallenberg [20] tells us that $Y_1^\epsilon \xrightarrow{d} B_1$ as $\epsilon \to 0$ if and only if, for each $h > 0$, $\int_{\|z\|\leq h} zz^T\nu_\epsilon(dz) \to I$, $\int_{\|z\|>h} z\nu_\epsilon(dz) \to 0$ and $\nu_\epsilon \to 0$ as $\epsilon \to 0$. The last condition is clear. For the first condition, observe that

$$\int_{\|z\|\leq h} zz^T\nu_\epsilon(dz) = \int_{E_h^\epsilon} zA_\epsilon^{-1}z^T\nu(dz) = A_\epsilon^{-1}\int_{E_h^\epsilon} zz^T\nu(dz),$$

where $E_h^\epsilon = \{z \in \mathbb{R}_0^d : \|(\beta(\epsilon)^T)^{-1}z\| \leq h\} \cap S_\epsilon$. Since $\|M_\epsilon\|_O \to 0$ and

$$E_h^\epsilon = \{\epsilon z \in \mathbb{R}_0^d : \|\epsilon\beta(\epsilon)^{-1}z\| \leq h\} \cap S_1 \sim S_\epsilon$$

as $\epsilon \to 0$, we get $A_\epsilon^{-1}\int_{E_h^\epsilon} zz^T\nu(dz) \to I$. For the second condition,

$$\int_{\|z\|>h} z\nu_\epsilon(dz) = \int_{F_h^\epsilon}(M_\epsilon^T)^{-1}z\nu(dz),$$

where $F_h^\epsilon = \{z \in \mathbb{R}_0^d : \|(\beta(\epsilon)^T)^{-1}z\| > h\} \cap S_\epsilon$. Since

$$\Big\| \int_{F_h^\epsilon} (\beta(\epsilon)^T)^{-1} z\nu(dz) \Big\| \le \int_{F_h^\epsilon} \|(M_\epsilon^T)^{-1}z\|\nu(dz) \le h\nu(F_h^\epsilon)$$

and $F_h^\epsilon = \{\epsilon z \in \mathbb{R}_0^d : \|\epsilon(\beta(\epsilon)^T)^{-1}z\| > h\} \cap S_1 \to \emptyset$ as $\epsilon \to 0$, we get $\int_{\|z\|>h} z\nu_\epsilon(dz) \to 0$. The proof is complete.

1.5 Malliavin Calculus and Clark-Ocone Formula

The Malliavin calculus is an infinite-dimensional differential calculus and is sometimes called the stochastic calculus of variations. Over the past few years, the Malliavin calculus have attracted a vast attention in mathematical finance. In this section, we will describe the Clark-Ocone formula of the Malliavin calculus for additive processes, following the representation of Løkka [27].

Let $\{X_t : t \in [0, T]\}$ be an additive process in $\mathbb{R}$ defined by

$$X_t = \gamma_t + G_t + \int_0^t \int_{\mathbb{R}_0} z(\varsigma - \varrho)(dz, ds),$$

where γ is some continuous function with $\gamma_0 = 0$, $\{G_t : t \in [0, T]\}$ is a centered Gaussian process in $\mathbb{R}$ with independent increments, $G_0 = 0$ and $E[G_t^2] = \int_0^t \sigma^2(s)ds$ for some nonnegative function σ, and ς is a Poisson random measure whose intensity measure ϱ satisfies, for $t \in [0, T]$,

$$\int_0^t \int_{\mathbb{R}_0} z^2 \varrho(dz, ds) < \infty.$$

Let ξ be a random variable on the filtered space $\mathbb{H} := L_2(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$. Since $\{G_\cdot, \{(\varsigma - \varrho)(dz, \cdot)\}\}$ constitutes a *basis for the predictable representation* for $\mathbb{H}$, we have the following martingale representation

$$\xi = E[\xi] + \int_0^T \phi_s dG_s + \int_0^T \int_{\mathbb{R}_0} \psi_{z,s}(\varsigma - \varrho)(dz, ds),$$

where ϕ_t and $\psi_{z,t}$ are predictable processes satisfying, respectively, $E[\int_0^T \phi_s^2 ds] < \infty$ and $E[\int_0^T \int_{\mathbb{R}_0} \psi_{z,s}^2 \varrho(dz, ds)] < \infty$. The following is called the *Itô isometry*,

$$E[\xi^2] = (E[\xi])^2 + \int_0^T E[\phi_s^2]\sigma(s)^2 ds + \int_0^T \int_{\mathbb{R}_0} E[\psi_{z,s}^2]\varrho(dz, ds). \tag{1.5.1}$$

On the other hand, by the chaos expansion, for every $\xi \in \mathbb{H}$, there exists a unique sequence of symmetric functions $\{g_n\}_{n \geq 0}$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(g_n)$$

where

$$I_n(g_n) := \sum_{(\alpha_1,\dots,\alpha_n) \in \mathcal{I}^n} \int_{[0,T]^n} g^{(\alpha_1,\dots,\alpha_n)}(t_1,\dots,t_n) dM_{t_1}^{\alpha_1} \otimes \dots \otimes dM_{t_n}^{\alpha_n},$$

with $\{M^\alpha\}_{\alpha \in \mathcal{I}} = \{G_\cdot, \{(\varsigma - \varrho)(dz, \cdot)\}\}$. Define $\mathbb{D}_{1,2} \subset L_2(\Omega)$ by

$$\mathbb{D}_{1,2} := \left\{ \xi = \sum_{n=0}^{\infty} I_n(g_n) : \sum_{n=1}^{\infty} n \cdot n! \|g_n\|^2 < \infty \right\}.$$

Moreover, for $\xi \in \mathbb{D}_{1,2}$, define the *Malliavin derivative operator*

$$D_{t,\alpha}\xi := \sum_{n=1}^{\infty} n I_{n-1}(g_n^\alpha(\cdot, t)).$$

Let D_t and $D_{z,t}$ denote the Malliavin derivative operators, respectively, with respect to G_t and $(\varsigma - \varrho)(dz, dt)$. It is known that $\phi_t = E[D_t\xi | \mathcal{F}_{t-}]$ and $\psi_{z,t} = E[D_{z,t}\xi | \mathcal{F}_{t-}]$. The quantity $E[\cdot | \mathcal{F}_{t-}]$ is called the *predictable projection.* The following is the *Clark-Ocone formula* for additive processes;

$$\xi = E[\xi] + \int_0^T E[D_s\xi | \mathcal{F}_{s-}] dG_s + \int_0^T \int_{\mathbb{R}_0} E[D_{z,s}\xi | \mathcal{F}_{s-}](\varsigma - \varrho)(dz, ds). \tag{1.5.2}$$

Moreover, if $\xi = f(X_T)$, we have

$$D_t\xi = f'(X_T) D_t X_T \quad \text{and} \quad D_{z,t}\xi = f(X_T + D_{z,t}X_T) - f(X_T), \tag{1.5.3}$$

where the first equality holds if f' exists on $\mathbb{R}$.

Remark 1.5.1. There is also a different form of the Clark-Ocone formula for Lévy processes introduced by Nualart and Schoutens [35]. Since it is not useful for our purpose, we do not introduce that.

1.6 Notes

Basic notations and definitions of Lévy processes and additive processes follow the book of Sato [44]. The terminology "infinitely divisible processes" was, to the best of the author's knowledge, first used by Maruyama [31]. Although not widely used at present, the name is natural.

Other than the Clark-Ocone formula for Lévy processes reviewed in Section 1.5, many Malliavin calculus formulas for Lévy processes are derived in DiNunno et al.[12], such as integration by parts and the Itô-Lévy-Skorohod isometry.

Chapter 2

Tempered Stable Processes

2.1 Preliminaries

Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$ and $\widehat{\mu}$ the characteristic function of μ. It is called *stable* if, for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}^d$ such that

$$\widehat{\mu}(y)^a = \widehat{\mu}(by)e^{i\langle c,y\rangle}. \tag{2.1.1}$$

Let $\{X_t : t \geq 0\}$ be a Lévy process with $\mathcal{L}(X_1) \sim \mu$ satisfying (2.1.1). Then, for every $a \geq 0$,

$$\{X_{at} : t \geq 0\} \stackrel{d}{=} \{bX_t + ct : t \geq 0\}.$$

This property is called *selfsimilarity.* Moreover, $\{X_t : t \geq 0\}$ is called a *stable process* if

$$\{X_{at} : t \geq 0\} \stackrel{d}{=} \{a^{1/\alpha}X_t + ct : t \geq 0\}.$$

where $0 < \alpha \leq 2$ and $c \in \mathbb{R}$. For example, the Brownian motion is a 2-stable process and the Cauchy process is a 1-stable process. The parameter α is called the *stability index.*

Let $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$, the unit sphere on $\mathbb{R}^d$. Then, the Lévy measure ν_α of an α-stable process can be written, in polar coordinates, as

$$\nu_\alpha(B) = \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_B(r\xi)\frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where λ is a finite non-zero measure on S^{d-1}. It is known that the $\int_{\|z\|\le 1}\|z\|\nu_\alpha(dz) < \infty$ if and only if $\alpha \in (0,1)$, and $\int_{\|z\|>1}\|z\|\nu_\alpha(dz) < \infty$ if and only if $\alpha \in (1,2)$. Moreover, $\nu_\alpha(\mathbb{R}_0^d) = \infty$. A non-trivial stable distribution on $\mathbb{R}^d$ has the characteristic function

$$\widehat{\mu}(y) = \begin{cases} \exp\left[-\int_{S^{d-1}} |\langle y,\xi\rangle|^\alpha \left(1 - i\tan\frac{\pi\alpha}{2}\mathrm{sgn}\langle y,\xi\rangle\right)\lambda(d\xi) + i\langle \tau, y\rangle\right], & \text{if } \alpha \neq 1, \\ \exp\left[-\int_{S^{d-1}} \left(|\langle y,\xi\rangle| + i\frac{2}{\pi}\langle y,\xi\rangle \ln|\langle y,\xi\rangle|\right)\lambda(d\xi) + i\langle \tau, y\rangle\right], & \text{if } \alpha = 1, \end{cases}$$

where $\tau \in \mathbb{R}^d$. When $d = 1$, the characteristic function reduces to

$$\widehat{\mu}(y) = \begin{cases} \exp\left[-c|y|^\alpha\left(1 - i\beta\,\mathrm{sgn}(y)\tan\frac{\pi\alpha}{2}\right) + iy\gamma\right], & \alpha \neq 1 \\ \exp\left[-c|y|\left(1 + i\beta\,\mathrm{sgn}(y)\frac{2}{\pi}\log|y|\right) + iy\gamma\right], & \alpha = 1, \end{cases}$$

where $\beta \in [-1,1]$, $c > 0$ and $\gamma \in \mathbb{R}$. The parameters β and c are called the *skewness parameter* and the *scale parameter*, respectively. Moreover, the collection of parameters (α,β,c,γ) is called the *stable law parameters*. A stable law generated by (α,β,c,γ) is often denoted by $S_\alpha(c,\beta,\gamma)$.

Let us here review some basic facts on stable distributions and processes. We denote the signed power by $a^{\langle p\rangle}$ for $a, p \in \mathbb{R}$, i.e. $a^{\langle p\rangle} := |a|^p\mathrm{sgn}(a)$.

(i) Let X_1 and X_2 be independent α-stable random variables with $\mathcal{L}(X_i) \sim S_\alpha(c_i,\beta_i,\gamma_i)$, $i = 1,2$, respectively. Then,

$$\mathcal{L}(X_1 + X_2) \sim S_\alpha(c,\beta,\gamma),$$

where

$$c = (c_1^\alpha + c_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1 c_1^\alpha + \beta_2 c_2^\alpha}{c_1^\alpha + c_2^\alpha}, \quad \gamma = \gamma_1 + \gamma_2.$$

(ii) Let X_3 and X_4 be iid α-stable random variables with $\mathcal{L}(X_i) \sim S_\alpha(c,\beta,\gamma)$ for $i = 3,4$. Then, for a, $b > 0$ and $c \in \mathbb{R}$,

$$\mathcal{L}(aX_3 + bX_4 + c) \sim S_\alpha(c(a^\alpha + b^\alpha)^{1/\alpha}, \beta, \gamma(a^\alpha + b^\alpha)^{1/\alpha} + c).$$

(iii) Let $\{X_t^\alpha : t \geq 0\}$ be an α-stable process where $\mathcal{L}(X_1^\alpha) \sim S_\alpha(c, \beta, \gamma)$ and fix $M \in \mathcal{B}([0, \infty))$. Then, for $f \in L^\alpha(M)$,

$$\mathcal{L}(\int_M f_s dX_s^\alpha) \sim S_\alpha(c', \beta', \gamma'),$$

where

$$c' = c(\int_M |f_s|^\alpha ds)^{1/\alpha}, \quad \beta' = \beta \frac{\int_M f_s^{\langle\alpha\rangle} ds}{\int_M |f_s|^\alpha ds}, \quad \gamma' = \gamma \int_M f_s ds.$$

Stable processes with a smaller stability index move mainly by big jumps while by small jumps if α is close to 2. In Figure 2.1, we give typical sample paths of symmetric α-stable processes $\{X_t^\alpha : t \geq 0\}$ with $\mathcal{L}(X_1^\alpha) \sim S_\alpha(1, 0, 0)$ for α=0.8, 1.4, 1.7 and 2.0. For detailed theoretical and simulation analysis of stable processes, see, for example, Sato [44], Samorodnitsky and Taqqu [45], and Janicki and Weron [18].

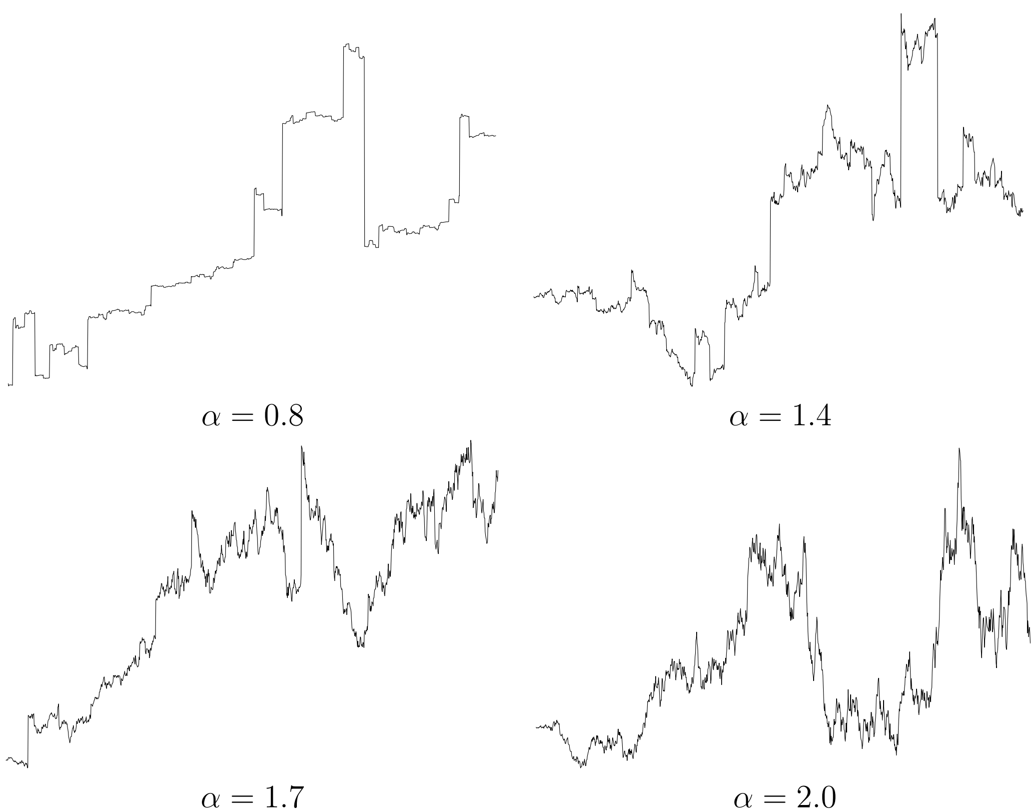


Figure 2.1: Typical sample paths of stable processes

In the physics literature, non-Gaussian α-stable processes are sometimes called

Lévy flights or *Lévy walks.* They have been widely used to better capture local spatiotemporal fractality observed in fluid dynamics, polymers and turbulent fluids. On the other hand, those observations still have finite variance unlike stable processes. Mantegna and Stanley [30] resolve this problem by truncating the tails of a density function f, i.e.

$$f_1(x) = \begin{cases} 0, & |x| > l, \\ c_1 f(x), & |x| \leq l, \end{cases} \tag{2.1.2}$$

where c_1 is a normalizing constant and $l > 0$ is the cutoff length. Stochastic processes with the density f_1 have finite second moment and thus belongs to the domain of attraction of Gaussian. They are termed *truncated Lévy flight* (TLF). Novikov [32] observes that the sum of independent TLF's converges very slowly to Gaussian distribution, and a huge number of a summation, in the order of $n = 10^4$, may be needed, in contrast to $n \simeq 10$ for most distributions.

A smooth cutoff of the Lévy flight is introduced by Koponen [23]. Instead of truncating tails of a distribution, this approach is based upon the exponential tempering of the Lévy density f (Radon-Nikodym derivative of the Lévy measure with respect to the Lebesgue measure), i.e.

$$f_2(x) = c_2 e^{-\lambda |x|} f(x), \tag{2.1.3}$$

where c_2 is a normalizing constant and $\lambda > 0$.

In the mathematical finance literature, a similar class of Lévy processes has recently been introduced by Carr, Geman, Madan and Yor [9]. This class is characterized by Lévy measure of the form

$$\nu(dz) = C|z|^{-Y-1}(e^{-G|z|}1_{\{z<0\}}(z) + e^{-M|z|}1_{\{z>0\}}(z))dz$$

with C, G, and $M \in \mathbb{R}^+$ and $Y \in (0, 2)$, and will be referred to as the CGMY class. They empirically observe that the CGMY processes nicely capture the dynamics of asset prices.

2.2 Tempered Stable Distributions

In the rest of this chapter, we study a more general and robust class of *tempered stable processes*, first introduced by Rosiński [41]. (See also Figueroa [14].) This class possesses many interesting features, for example;

(i) they behave like an α-stable process in short time periods,

(ii) they are approximately Gaussian in the long run,

(iii) unlike α-stable processes, they have all moments finite under a suitable condition.

We begin with the definition of the tempered stable distribution.

Definition 2.2.1. *A probability measure μ on $\mathbb{R}^d$ is called tempered stable if it is infinitely divisible without Gaussian component and with Lévy measure ν on $\mathbb{R}^d_0$ of the form*

$$\nu(B) = \int_{\mathbb{R}^d_0} \int_0^\infty 1_B(sx) s^{-\alpha-1} e^{-s} ds\, \rho(dx), \quad B \in \mathcal{B}(\mathbb{R}^d_0) \tag{2.2.1}$$

where $\alpha \in (0,2)$ and ρ is a σ-finite Borel measure on $\mathbb{R}^d_0$ such that

$$\int_{\mathbb{R}^d_0} \|x\|^\alpha \rho(dx) < \infty. \tag{2.2.2}$$

Consider a Lévy measure without the exponential term e^{-s} of (2.2.1). It turns out to be the Lévy measure of α-stable distributions. Indeed,

$$\int_{\mathbb{R}^d_0} \int_0^\infty 1_B(sx) s^{-\alpha-1} ds\, \rho(dx) = \int_{S^{d-1}} \sigma_\alpha(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{\alpha+1}},$$

where

$$\sigma_\alpha(C) := \frac{\int_{\mathbb{R}^d_0} 1_C(x/\|x\|)\|x\|^\alpha \rho(dx)}{\int_{\mathbb{R}^d_0} \|x\|^\alpha \rho(dx)}, \quad C \in \mathcal{B}(S^{d-1}).$$

For convenience in later discussions, we set

$$\nu_\alpha(B) := \int_{\mathbb{R}^d_0} \int_0^\infty 1_B(sx) s^{-\alpha-1} ds\, \rho(dx), \quad B \in \mathcal{B}(\mathbb{R}^d_0), \tag{2.2.3}$$

and call an α-stable process induced by ν_α the *corresponding* α-stable process.

Remark 2.2.2. Notice that the condition (2.2.2) is necessary and sufficient so that the stable Lévy measure ν_α given by (2.2.3) is well defined as a Lévy measure. This can be seen by

$$\int_{\mathbb{R}_0^d} (\|z\|^2 \wedge 1)\nu_\alpha(dz) = \frac{2}{\alpha(2-\alpha)} \int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx).$$

This implies that (2.2.2) is not necessary for ν to be a Lévy measure because $\nu(B) < \nu_\alpha(B)$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$. Let us see that a necessary and sufficient condition is given by

$$\int_{\mathbb{R}_0^d} (\|x\|^2 \wedge \|x\|^\alpha)\rho(dx) < \infty. \tag{2.2.4}$$

First, observe that

$$\begin{aligned}\int_{\mathbb{R}_0^d} (\|z\|^2 \wedge 1)\nu(dz) &= \int_{\mathbb{R}_0^d} \left(\|x\|^2 \int_0^{1/\|x\|} s^{-\alpha+1}e^{-s}ds + \int_{1/\|x\|}^\infty s^{-\alpha-1}e^{-s}ds \right) \rho(dx) \\ &= \int_{\mathbb{R}_0^d} \|x\|^\alpha \left(\int_0^1 s^{-\alpha+1}e^{-s/\|x\|}ds + \int_1^\infty s^{-\alpha-1}e^{-s/\|x\|}ds \right) \rho(dx).\end{aligned}$$

We will consider two domains of the measure ρ; $\{\|x\| \leq 1\}$ and $\{\|x\| > 1\}$. On $\{\|x\| > 1\}$, $\int_0^1 s^{-\alpha+1}e^{-s/\|x\|}ds$ is bounded from above by $1/(2-\alpha)$ and from below by $\int_0^1 s^{-\alpha+1}e^{-s}ds$, and $\int_1^\infty s^{-\alpha-1}e^{-s/\|x\|}ds$ is bounded from above by $1/\alpha$ and from below by $\int_1^\infty s^{-\alpha-1}e^{-s}ds$. Therefore, we need $\int_{\|x\|>1} \|x\|^\alpha \rho(dx) < \infty$. For the domain $\{\|x\| \leq 1\}$, we have that $\int_0^{1/\|x\|} s^{-\alpha+1}e^{-s}ds$ is bounded from above by $\Gamma(2-\alpha)$ and from below by $\int_0^1 s^{-\alpha+1}e^{-s}$, and that $\int_{1/\|x\|}^\infty s^{-\alpha-1}e^{-s}ds = o(\|x\|^2)$ as $\|x\| \to 0$. Hence, we need $\int_{\|x\|\leq 1} \|x\|^2 \rho(dx) < \infty$. Putting those together, we get (2.2.4).

We say that a function f on $(0,\infty)$ is *completely monotone* if $(-1)^n f^{(n)} \geq 0$ for all $n \in \mathbb{N}$. The tempered stable distribution can also be characterized by the Lévy measure in polar coordinates.

Proposition 2.2.3. *ν is the Lévy measure of a tempered stable distribution μ on $\mathbb{R}^d$ if and only if in polar coordinates it has the form*

$$\nu(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) q(r,\xi) \frac{dr}{r^{\alpha+1}}, \qquad B \in \mathcal{B}(\mathbb{R}_0^d) \tag{2.2.5}$$

where $q : (0,\infty) \times S^{d-1} \mapsto (0,\infty)$ is a Borel function such that $q(\cdot,\xi)$ is completely monotone with $\lim_{r\to\infty} q(r,\xi) = 0$ for every $\xi \in S^{d-1}$, and σ is a probability measure on S^{d-1} such that

$$\int_{S^{d-1}} q(0+,\xi)\sigma(d\xi) < \infty. \tag{2.2.6}$$

Proof. By Bernstein's theorem, $q(\cdot,\xi)$ is completely monotone in ξ if and only if it is uniquely written as

$$q(r,\xi) = \int_{(0,\infty)} e^{-ry}\psi_\xi(dy),$$

with some measure ψ_ξ on $(0,\infty)$ such that the integral is well defined. Then, for $B \in \mathcal{B}(\mathbb{R}_0^d)$,

$$\begin{aligned}
\nu(B) &= \int_{S^{d-1}} \int_0^\infty 1_B(r\xi) \int_{(0,\infty)} r^{-\alpha-1} e^{-ry}\psi_\xi(dy) dr \sigma(d\xi) \\
&= \int_{S^{d-1}} \int_{(0,\infty)} \left(\int_0^\infty 1_B(s \cdot y^{-1}\xi) s^{-\alpha-1} e^{-s} ds \right) y^\alpha \psi_\xi(dy)\sigma(d\xi) \\
&= \int_{\mathbb{R}_0^d} \int_0^\infty 1_B(sx) s^{-\alpha-1} e^{-s} ds \rho(dx),
\end{aligned}$$

where

$$\rho(dr\, d\xi) = r^{-\alpha}\widetilde{\psi}_\xi(dr)\sigma(d\xi), \quad (r,\xi) \in (0,\infty) \times S^{d-1}$$

with $\widetilde{\psi}_\xi(B) := \psi_\xi(B^{-1})$ for $B \in \mathcal{B}(0,\infty)$. Finally, (2.2.6) is obtained by

$$\int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) = \int_{(0,\infty) S^{d-1}} \psi_\xi(dr)\sigma(d\xi) = \int_{S^{d-1}} q(0+,\xi)\sigma(d\xi),$$

which concludes the proof.

Remark 2.2.4. Notice that μ becomes the corresponding α-stable distribution if $q(r,\xi) \equiv q(0+,\xi)$ for every $\xi \in S^{d-1}$, i.e., its Lévy measure is given by

$$\nu_\alpha(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) q(0+,\xi) \frac{dr}{r^{\alpha+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d). \tag{2.2.7}$$

Moreover, ν is some Lévy measure if and only if

$$\int_{S^{d-1}} \left(\int_{(0,1]} \psi_\xi(dr) + \int_{(1,\infty)} r^{\alpha-2}\psi_\xi(dr) \right) \sigma(d\xi) < \infty,$$

which is equivalent to (2.2.4). Moreover, the structure (2.2.5) implies that a tempered stable distribution μ are selfdecomposable. This can also be shown as follows. Define a transformation T_r of measures ν on $\mathbb{R}^d$ by $(T_r\nu)(B) = \nu(r^{-1}B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Then, by Theorem 15.8 of Sato [44], since $T_b\nu \geq \nu$ for $b > 1$, μ is selfdecomposable. In fact,

$$\begin{aligned} T_b\nu(B) &= \nu(b^{-1}B) = \int_{\mathbb{R}_0^d}\int_0^\infty 1_{b^{-1}B}(sx)s^{-\alpha-1}e^{-s}ds\ \rho(dx) \\ &= b^\alpha \int_{\mathbb{R}_0^d}\int_0^\infty 1_B(sx)s^{-\alpha-1}e^{-b^{-1}s}ds\ \rho(dx) \\ &\geq \int_{\mathbb{R}_0^d}\int_0^\infty 1_B(sx)s^{-\alpha-1}e^{-s}ds\ \rho(dx) = \nu(B). \end{aligned}$$

In addition, a tempered stable distribution μ is absolutely continuous by Theorem 27.13 of Sato [44].

The following is a direct consequence from Orey [37].

Corollary 2.2.5. *A nondegenerate tempered stable distribution μ on $\mathbb{R}$ has a density of class C^∞ and all derivatives of the density tend to 0 as $|x| \to \infty$.*

Proof. By Orey [37], it is enough to check $\liminf_{\epsilon\downarrow 0}\epsilon^{a-2}\int_{[-\epsilon,\epsilon]} z^2\nu(dz) > 0$ for some $a \in (0,2)$. Setting $a = \alpha$, we have

$$\begin{aligned} \frac{\int_{[-\epsilon,\epsilon]} z^2\nu(dz)}{\epsilon^{2-\alpha}} &= \frac{\epsilon^{\alpha-2}}{2-\alpha}\int_{\mathbb{R}_0} x^2\left(e^{-\epsilon|x|}\left(\frac{\epsilon}{|x|}\right)^{2-\alpha} + \int_0^{\epsilon/|x|} e^{-s}s^{2-\alpha}\right)\rho(dx) \\ &\geq \frac{1}{2-\alpha}\int_{\mathbb{R}_0} |x|^\alpha e^{-\epsilon/|x|}\rho(dx). \end{aligned}$$

Thus, we get

$$\liminf_{\epsilon\downarrow 0}\frac{\int_{[-\epsilon,\epsilon]} z^2\nu(dz)}{\epsilon^{2-\alpha}} \geq \frac{1}{2-\alpha}\int_{\mathbb{R}_0}|x|^\alpha\rho(dx) > 0,$$

which concludes the proof.

Let us derive the characteristic function of a tempered stable distribution. Here, we write $\ln^+(a) := \ln a$ if $a > 1$, or 0 otherwise.

Proposition 2.2.6. *Let μ be a tempered α-stable distribution defined as above. When $\alpha = 1$, assume further that*

$$\int_{\mathbb{R}_0^d} \|x\|(1 + \ln^+ \|x\|)\rho(dx) < \infty. \tag{2.2.8}$$

Then, the characteristic function $\widehat{\mu}$ of a tempered stable distribution μ with Lévy measure (2.2.1) is given by

$$\widehat{\mu}(y) = \exp\left[k_\alpha \int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\rho(dx) + i\langle y, \gamma\rangle\right] \tag{2.2.9}$$

where $\gamma \in \mathbb{R}^d$ and

$$\psi_\alpha(s) = \begin{cases} \Gamma(-\alpha)((1-is)^\alpha - 1), & \text{if } 0 < \alpha < 1 \\ (1-is)\ln(1-is) + is, & \text{if } \alpha = 1 \\ \Gamma(-\alpha)((1-is)^\alpha - 1 + i\alpha s), & \text{if } 1 < \alpha < 2. \end{cases} \tag{2.2.10}$$

Proof. Clearly, $\nu(B) \leq \nu_\alpha(B)$, for $B \in \mathcal{B}(\mathbb{R}_0^d)$. Moreover, when $\alpha = 1$, (2.2.8) implies $\int_{\|z\|>1} \|z\|\nu(dz) < \infty$. In fact,

$$\begin{aligned} \int_{\|z\|>1} \|z\|\nu(dz) &= \int_{\mathbb{R}_0^d} \left(e^{-\|x\|^{-1}} \ln\|x\| + \int_{\|x\|^{-1}}^{\infty} e^{-s}\ln s\, ds\right) \|x\|\rho(dx) \\ &\leq \int_{\mathbb{R}_0^d} (1 + \ln^+\|x\|)\|x\|\rho(dx). \end{aligned}$$

Hence, when $\alpha \in (0, 1)$, we have $\int_{\|z\|\leq 1} \|z\|\nu(dz) < \infty$ and thus it suffices to consider $\exp\left[\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1)\nu(dz)\right]$. Similarly, when $\alpha \in [1, 2)$, we have $\int_{\|z\|>1} \|z\|\nu(dz) < \infty$ and so we will consider $\exp\left[\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1)\nu(dz)\right]$.

For $0 < \alpha < 1$, we have

$$\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1)\nu(dz) = \int_{\mathbb{R}_0^d} \int_0^\infty (e^{is\langle y,x\rangle} - 1)s^{-\alpha-1}e^{-s}ds\ \rho(dx).$$

Then,

$$\int_0^\infty e^{is\langle y,x\rangle} s^{-\alpha-1}e^{-s}ds = (1 - i\langle y, x\rangle)^\alpha \int_0^\infty u^{-\alpha-1}e^{-u}du = \Gamma(-\alpha)(1 - i\langle y, x\rangle)^\alpha$$

and $\int_0^\infty s^{-\alpha-1}e^{-s}ds = \Gamma(-\alpha)$ give the result. For $1 < \alpha < 2$, we evaluate

$$\int_{\mathbb{R}_0^d}\int_0^\infty (e^{is\langle y,x\rangle} - 1 - is\langle y,x\rangle)s^{-\alpha-1}e^{-s}ds\ \rho(dx).$$

The first and second terms are same as in (i). For the third term,

$$\int_0^\infty is\langle y,x\rangle s^{-\alpha-1}e^{-s}ds = i\langle y,x\rangle\Gamma(-\alpha+1) = -i\alpha\langle y,x\rangle\Gamma(-\alpha).$$

Hence, we get

$$\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle)\nu(dz) = \Gamma(-\alpha)\int_{\mathbb{R}_0^d}((1 - i\langle y,x\rangle)^\alpha - 1 + i\alpha\langle y,x\rangle)\rho(dx).$$

Finally, for the case of $\alpha = 1$, we have

$$k_\alpha\psi_\alpha(s) = -\frac{\Gamma(2-\alpha)}{\alpha}\frac{(1-is)^\alpha - 1 + i\alpha s}{1-\alpha} \to \psi_1(s),$$

as $\alpha \downarrow 1$, which concludes the proof.

We show in the following that a tempered stable distribution is uniquely characterized by three parameters α, ρ and γ.

Theorem 2.2.7. *In the above notation, if*

$$\int_{\mathbb{R}_0^d}\psi_\alpha(\langle y,x\rangle)\rho_1(dx) = \int_{\mathbb{R}_0^d}\psi_\alpha(\langle y,x\rangle)\rho_2(dx), \quad y \in \mathbb{R}^d, \tag{2.2.11}$$

where ρ_j are Borel measures on $\mathbb{R}^d$ with $\int_{\mathbb{R}_0^d}||x||^\alpha\rho_j(dx) < \infty, j = 1,2$, then $\rho_1 = \rho_2$.

Proof. By Proposition 2.2.6,

$$\widehat{\mu}_j(y) := \exp\left[k_\alpha\int_{\mathbb{R}_0^d}\psi_\alpha(\langle y,x\rangle)\rho_j(dx)\right], \quad j = 1,2,$$

are characteristic functions of a common infinitely divisible distribution with no Gaussian component. Denote by ν_j the Lévy measure of μ_j for $j = 1,2$. Then, the Lévy-Khintchine representation tells us that $\nu_1 = \nu_2$. Hence, for every $B \in \mathcal{B}(\mathbb{R}_0^d)$,

$$(\nu_1 - \nu_2)(B) = \int_{\mathbb{R}_0^d}\int_0^\infty 1_B(sx)s^{-\alpha-1}e^{-s}ds(\rho_1 - \rho_2)(dx) = 0.$$

Since this equality holds for every $B \in \mathcal{B}(\mathbb{R}_0^d)$ and $s^{-\alpha-1}e^{-s} > 0$, we get $\rho_1 = \rho_2$. The proof is complete.

Let us call the measure ρ on $\mathbb{R}^d_0$ an *inner measure*. For convenience, when (2.2.9) holds, we write

$$\mu \sim TS(\alpha, \rho\,; \gamma). \tag{2.2.12}$$

Clearly, sum of independent tempered stable random variables with a common α is again a tempered stable random variable. That is, let $\{X_i\}_{i=1}^k$ be a sequence of independent tempered stable random vectors in $\mathbb{R}^d$ with $\mathcal{L}(X_i) \sim TS(\alpha, \rho_i; \gamma_i)$ for each i. Then, by Proposition 2.2.6, we have $\mathcal{L}(\sum_{i=1}^k X_i) \sim TS(\alpha, \sum_{i=1}^k \rho_i; \sum_{i=1}^k \gamma_i)$.

We will derive conditions for the moments of tempered stable distributions.

Proposition 2.2.8. *Let μ be a tempered stable distribution on $\mathbb{R}^d$ with Lévy measure (2.2.1). Then,*

(i) $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ for every $p \in (0, \alpha)$,

(ii) $\int_{\mathbb{R}^d} \|x\|^\alpha \mu(dx) < \infty$ if and only if

$$\int_{\|x\|>1} \|x\|^\alpha \ln \|x\| \rho(dx) < \infty,$$

(iii) $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ for $p > \alpha$ if and only if

$$\int_{\|x\|>1} \|x\|^p \rho(dx) < \infty.$$

(iv) if $\rho(\{x \in \mathbb{R}^d_0 : \|x\| > \epsilon\}) = 0$ for some $\epsilon > 0$, then for every $\theta \in (0, \epsilon^{-1})$,

$$\int_{\mathbb{R}^d} \exp(\theta \|x\|) \mu(dx) < \infty.$$

Proof. (i) This is clear because $\nu(B) \le \nu_\alpha(B)$ for every $B \in \mathcal{B}(\mathbb{R}^d_0)$.

(ii), (iii) By Theorem 1.2.2, we will consider the finiteness of $\int_{\|z\|>1} \|z\|^p \nu(dz)$. We have

$$\begin{aligned}
\int_{\|z\|>1} \|z\|^p \nu(dz) &= \int_{\mathbb{R}^d_0} \|x\|^p \int_{1/\|x\|}^\infty s^{p-\alpha-1} e^{-s} ds\ \rho(dx) \\
&= \int_{\|x\|>1} \|x\|^p \int_{1/\|x\|}^\infty s^{p-\alpha-1} e^{-s} ds\ \rho(dx) \\
&\quad + \int_{\|x\|\le 1} \|x\|^p \int_{1/\|x\|}^\infty s^{p-\alpha-1} e^{-s} ds\ \rho(dx),
\end{aligned}$$

where the second term is finite because

$$\int_{\|x\|\leq 1} \|x\|^p \int_{1/||x||}^{\infty} s^{p-\alpha-1} e^{-s} ds \ \rho(dx) \leq \int_1^{\infty} s^{p-\alpha-1} e^{-s} ds \int_{\|x\|\leq 1} \|x\|^{\alpha} \rho(dx) < \infty.$$

Therefore, it is enough to show that $\int_{\|x\|>1} \|x\|^{\alpha} \int_{1/\|x\|}^{\infty} s^{p-\alpha-1} e^{-s} ds \ \rho(dx) < \infty$ is equivalent to $\int_{\|x\|>1} \|x\|^{\alpha} \ln \|x\| \rho(dx) < \infty$ when $p = \alpha$, or to $\int_{\|x\|>1} \|x\|^p \rho(dx) < \infty$ when $p > \alpha$. The case $p > \alpha$ is clear because on $\{\|x\| > 1\}$, $\int_{1/\|x\|}^{\infty} s^{p-\alpha-1} e^{-s} ds$ is bounded from below by $\int_1^{\infty} s^{p-\alpha-1} e^{-s} ds$ and from above by $\Gamma(p-\alpha)$. For the case $p = \alpha$, $\int_{\|x\|^{-1}}^{\infty} s^{-1} e^{-s} ds$ on $\{\|x\| > 1\}$ is bounded from below by $0.5 \ln \|x\|$ and from above by $\int_1^{\infty} s^{-1} e^{-s} ds + \ln \|x\|$. The proof is complete.

(iv) Fix $\epsilon > 0$ such that $\rho(\{x \in \mathbb{R}_0^d : ||x|| > \epsilon\}) = 0$. Since $e^{\theta\|x\|}$ is submultiplicative for $\theta \in (0, \epsilon^{-1})$, $\int_{\|z\|>1} \exp(\theta\|z\|)\nu(dz) < \infty$ is equivalent to $\int_{\mathbb{R}^d} \exp(\theta\|x\|)\mu(dx) < \infty$ by Theorem 1.2.2. We have

$$\begin{aligned}
\int_{\|z\|>1} \exp(\theta\|z\|)\nu(dz) &= \iint_{s\|x\|>1} \exp(\theta s\|x\|) s^{-\alpha-1} e^{-s} ds \ \rho(dx) \\
&< \iint_{s\|x\|>1} s^{-\alpha-1} \exp(\epsilon(\theta - 1/\epsilon)s) ds \ \rho(dx) \\
&= (\theta\epsilon - 1)^{\alpha+1} \iint_{s\|x\|>1-\theta\epsilon} s^{-\alpha-1} e^{-s} ds \rho(dx) \\
&= (\theta\epsilon - 1)^{\alpha+1} \nu(\{z \in \mathbb{R}_0^d : \|z\| > 1 - \theta\epsilon\}) < \infty,
\end{aligned}$$

where the first inequality holds by $\rho(\{x \in \mathbb{R}_0^d : \|x\| > \epsilon\}) = 0$, and the condition $\theta \in (0, \epsilon^{-1})$ gives the second equality while the finiteness of $\nu(\{z \in \mathbb{R}_0^d : \|z\| > 1-\theta\epsilon\})$ follows from the definition of the Lévy measure, and again, $\theta \in (0, \epsilon^{-1})$. The proof is complete.

Proposition 2.2.8 (iv) reads that tempered stable distributions may have all finite moments. Recall that non-Gaussian α-stable distributions do not even possess finite second moment.

2.3 Tempered Stable Processes

Definition 2.3.1. *A Lévy process $\{X_t : t \geq 0\}$ in $\mathbb{R}^d$ is called a tempered stable process, if $\mathcal{L}(X_1) \sim TS(\alpha, \rho; \gamma)$. Then, we will write $\{X_t : t \geq 0\} \sim TS(\alpha, \rho; \gamma)$.*

We have seen that sum of independent tempered stable random variables with a common α is again tempered stable. Similarly, a time-changed and/or scaled tempered stable process is again a tempered stable process. Using the notation of Proposition 2.2.6, for each $a \in \mathbb{R}_0$, $b > 0$, and $t \geq 0$,

$$\begin{aligned}
\widehat{\mu}_{aX_{bt}}(y) &= \exp\left[bt\left(k_\alpha \int_{\mathbb{R}_0^d} \psi_\alpha(\langle ay, x\rangle)\rho(dx) + i\langle ay, \gamma\rangle\right)\right] \\
&= \exp\left[t\left(k_\alpha \int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, b\,(T_a\rho)(dx) + i\langle y, ab\gamma\rangle\right)\right].
\end{aligned}$$

Since $\int_{\mathbb{R}_0^d} \|x\|^\alpha b(T_a\rho)(dx) = |a|^\alpha b \int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) < \infty$, the measure $b(T_a\rho)$ is well defined as an inner measure. This shows that $\{aX_{bt} : t \geq 0\} \sim TS(\alpha, b(T_a\rho); ab\gamma)$. A natural further question will be whether or not a stochastic integral (of a suitable deterministic integrand) with respect to a tempered stable process has a tempered stable law. It is well known that a stochastic integral with respect to an α-stable process where $\alpha \in (0, 2]$ has a stable law with the same α. Let f be a locally bounded and measurable function on $[0, \infty)$ and $X_t := (X_t^1, \ldots, X_t^d) \in \mathbb{R}^d$. We denote a component-wise stochastic integral by, for every $t \geq 0$,

$$\int_0^t f(s) dX_s := \left(\int_0^t f(s) dX_s^1, \ldots, \int_0^t f(s) dX_s^d\right).$$

(This should not be confused with the following stochastic integral

$$\int_0^t \langle g(s), dX_s\rangle = \sum_{i=1}^d \int_0^t g_i(s) dX_s^i,$$

for a locally bounded and measurable $g = (g_1, \ldots, g_d) : [0, \infty) \to \mathbb{R}^d$.)

Proposition 2.3.2. *Let $\{X_t : t \geq 0\}$ be a tempered stable process in $\mathbb{R}^d$ with $TS(\alpha, \rho; 0)$. Then, for $f \in L^\alpha([0, t])$, the stochastic integral $\int_0^t f(s) dX_s$ has a tempered* *stable law* $TS(\alpha, \eta_t; 0)$*, where* $\eta_t = G \circ J_t$ *with* $G(ds, dx) := ds\rho(dx)$*, and*

$$J_t(z) := \{(s, x) \in [0, t] \times \mathbb{R}_0^d : xf(s) = z\}. \tag{2.3.1}$$

Proof. Fix $t > 0$. There always exists a sequence of step functions $\{f^{(n)}\}_{n\geq 1} \in L^\alpha([0,t])$ such that $f^{(n)} \to f$ a.e. on $[0,t]$ and $|f^{(n)}(s)| \leq |f(s)|$ for every $n \geq 1$ and $s \in [0,t]$. Then, by the linearity of stochastic integrals, we have

$$\int_0^t f^{(m)}(s)dX_s - \int_0^t f^{(n)}(s)dX_s = \int_0^t (f^{(m)}(s) - f^{(n)}(s))dX_s.$$

By the independence of increments of Lévy processes, we get

$$\begin{aligned} E[e^{i\langle y, \int_0^t (f^{(m)}(s) - f^{(n)}(s))dX_s\rangle}] &= \exp\left[\int_0^t \int_{\mathbb{R}_0^d} \psi_\alpha((f^{(m)}(s) - f^{(n)}(s))\langle y, x\rangle)\rho(dx)ds\right] \\ &= \exp\left[\int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, \eta_t^{f^{(m)}, f^{(n)}}(dx)\right], \end{aligned}$$

where $\eta_t^{f,g} = G \circ J_t^{f,g}$ with

$$J_t^{f,g}(B) = \left\{(s, x) \in [0,t] \times \mathbb{R}_0^d : x(f(s) - g(s)) \in B\right\}, \quad B \in \mathcal{B}(\mathbb{R}_0^d).$$

Here, for every $m, n \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}_0^d} \|x\|^\alpha \eta_t^{f^{(m)}, f^{(n)}}(dx) &= \int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) \int_0^t |f^{(m)}(s) - f^{(n)}(s)|^\alpha ds \\ &\leq 2^\alpha \int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) \int_0^t |f(s)|^\alpha ds < \infty. \end{aligned}$$

Hence, for every $m, n \geq 0$, the measure $\eta_t^{f^{(m)}, f^{(n)}}$ is well defined as an inner measure and thus

$$\mathcal{L}\left(\int_0^t (f^{(m)}(s) - f^{(n)}(s))dX_s\right) \sim TS(\alpha, \eta_t^{f^{(m)}, f^{(n)}}; 0).$$

By Theorem 2.2.7 and $\lim_{m,n\to\infty} \eta_t^{f^{(m)}, f^{(n)}}(B) \to 0$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$, we get

$$\mathcal{L}\left(\lim_{m,n\to\infty} \int_0^t (f^{(m)}(s) - f^{(n)}(s))dX_s\right) \sim TS(\alpha, 0; 0),$$

which is a degenerate random variable. It remains to show that the convergence in probability does not depend on the choice of a sequence of step functions. Let

$\{f^{(n)}\}_{n\geq 1} \in L^\alpha([0,t])$ and $\{g^{(n)}\}_{n\geq 1} \in L^\alpha([0,t])$ be sequences of step functions such that $f^{(n)} \to f$, $g^{(n)} \to g$ a.e. and $|f^{(n)}(s)| \leq |f(s)|$, $|g^{(n)}(s)| \leq |f(s)|$ for every $s \in [0,t]$. Then, similarly, we have $\mathcal{L}(\int_0^t (f^{(n)}(s) - g^{(n)}(s))dX_s) \sim TS(\alpha, \eta_t^{f^{(n)},g^{(n)}})$ and $\eta_t^{f^{(n)},g^{(n)}}(B) \to 0$ for $B \in \mathcal{B}(\mathbb{R}_0^d)$. This shows that $\int_0^t f^{(n)}(s)dX_s - \int_0^t g^{(n)}(s)dX_s \to 0$ in probability. Therefore, $\int_0^t f^{(n)}(s)dX_s$ converges in probability to $\int_0^t f(s)dX_s$ as $n \to \infty$. Since the convergence in probability implies the convergence in law, we get

$$E[e^{i\langle y, \int_0^t f(s)dX_s\rangle}] = \exp\left[\int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\eta_t(dx)\right],$$

which prove the claim.

In the following, we will show that a stochastic integral with respect to a tempered stable process enjoys the finiteness of an exponential moment under suitable condition.

Corollary 2.3.3. *Let $\{X_t : t \geq 0\}$ be a tempered stable process with $TS(\alpha, \rho; 0)$ and $f \in L^\alpha(\mathbb{R}^+)$. Then, if $\rho(\{x : ||x|| > \epsilon\}) = 0$ and $|f| < C < \infty$ for some $\epsilon > 0$ and $C > 0$, then the integral $\int_{\mathbb{R}^+} f_s dX_s$ has finite exponential moment with exponent $(0, (\epsilon C)^{-1})$.*

Proof. By Proposition 2.3.2, $\int_{\mathbb{R}^+} f_s dX_s$ has a tempered stable law. Thus, by Proposition 2.2.8, it suffices to show that its inner measure η defined in (2.3.2) has compact support. This is equivalent for J given by (2.3.1) to have compact support. But this is clearly so because ρ has compact support and f is bounded.

In the next theorem, we will derive the short-time behavior of tempered stable processes. It says that a tempered stable process behaves like exhibiting selfsimilarity with the same stability index in short period of time. Recall that $\xrightarrow{d}$ denotes the convergence of finite dimensional distributions.

Theorem 2.3.4. *Let $\{X_t : t \geq 0\}$ be a tempered stable Lévy process in $\mathbb{R}^d$ with $TS(\alpha, \rho; 0)$. If $\alpha \neq 1$, then*

$$\{h^{-1/\alpha} X_{ht} : t \geq 0\} \xrightarrow{d} \{Y_t : t \geq 0\} \quad as \quad h \to 0$$

where $\{Y_t : t \geq 0\}$ *is an* α*-stable Lévy process with characteristic function*

$$E[e^{i\langle y, Y_t\rangle}] = \exp\left[-tc_\alpha \int_{\mathbb{R}_0^d} |\langle y, x\rangle|^\alpha (1 - i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle y, x\rangle)\rho(dx)\right] \tag{2.3.2}$$

where $c_\alpha = \Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)$. *If* $\alpha = 1$, *then*

$$\{h^{-1}(X_{ht} - k_h t) : t \geq 0\} \xrightarrow{d} \{Y_t : t \geq 0\} \qquad as \quad h \to 0,$$

where $k_h = (1 + \ln h)\int_{\mathbb{R}_0^d} x\rho(dx)$, *and* $\{Y_t : t \geq 0\}$ *is a 1-stable Lévy process with characteristic function*

$$E[e^{i\langle y, Y_1\rangle}] = \exp\left[-\int_{\mathbb{R}_0^d}\left(\frac{\pi}{2}|\langle y, x\rangle| + i\langle y, x\rangle \ln|\langle y, x\rangle|\right)\rho(dx) + i\int_{\mathbb{R}_0^d}\langle y, x\rangle\rho(dx)\right], \tag{2.3.3}$$

with the usual convention $z\ln z = 0$ *if* $z = 0$.

Proof. Since for each $h > 0$, $\{h^{-1/\alpha}X_{ht} : t \geq 0\}$ is a Lévy process, it suffices to show that $h^{-1/\alpha}X_h \xrightarrow{d} Y_1$ as $h \to 0$. For $\alpha \neq 1$, set

$$\phi_\alpha(s) = \begin{cases} e^{is} - 1, & \text{if } 0 < \alpha < 1 \\ e^{is} - 1 - is, & \text{if } 1 < \alpha < 2. \end{cases}$$

By the change of variable $u = h^{-1/\alpha}s$, we get

$$\begin{aligned} \ln E[e^{i\langle y, h^{-1/\alpha}X_h\rangle}] &= h\int_{\mathbb{R}_0^d}\int_0^\infty \phi_\alpha(h^{-1/\alpha}s\langle y, x\rangle)s^{-\alpha-1}e^{-s}ds\ \rho(dx) \\ &= \int_{\mathbb{R}_0^d}\int_0^\infty \phi_\alpha(u\langle y, x\rangle)u^{-\alpha-1}e^{-h^{1/\alpha}u}du\ \rho(dx) \\ &\to \int_{\mathbb{R}_0^d}\int_0^\infty \phi_\alpha(u\langle y, x\rangle)u^{-\alpha-1}du\ \rho(dx) \qquad \text{as} \quad h \to 0, \end{aligned}$$

provided that the limit and the integral can be interchanged. The last term is equal to the right-hand side of (2.3.2) for $\alpha \neq 1$. If $\alpha = 1$, we have

$$E[e^{i\langle y, h^{-1}(X_h - k_h)\rangle}] = \exp\left[\int_{\mathbb{R}_0^d}\phi_1(h, \langle y, x\rangle)\rho(dx)\right],$$

where $\phi_1(h,s) = (h-is)\ln(h-is) - h\ln h + is$. The result holds by

$$\lim_{h\to 0} \phi_1(h,s) = -is\ln(-is) + is = -\left(\frac{\pi}{2}|s| + is\ln|s|\right) + is, \quad s \in \mathbb{R},$$

provided that the passage to the limit is justified. It now remains to justify the interchange of limits with integrals. If $\alpha \in (0,1)$, for each $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d_0$, and $h > 0$, we have

$$\begin{aligned}
\left|\int_0^\infty \phi_\alpha(u\langle y,x\rangle)u^{-\alpha-1}e^{-h^{-1/\alpha}u}du\right| &\le \int_0^\infty |e^{iu\langle y,x\rangle} - 1|u^{-\alpha-1}du \\
&\le \int_0^\infty (|u\langle y,x\rangle| \wedge 2)u^{-\alpha-1}du = |\langle y,x\rangle|^\alpha \int_0^\infty (s\wedge 2)s^{-\alpha-1}ds \\
&\le \|y\|^\alpha\|x\|^\alpha \int_0^\infty (s\wedge 2)s^{-\alpha-1}ds,
\end{aligned}$$

where the equality holds by the change of variables $s = u|\langle y,s\rangle|$, and $\int_0^\infty (s \wedge 2)s^{-\alpha-1}ds < \infty$. Similarly, If $\alpha \in (1,2)$, we have

$$\begin{aligned}
\left|\int_0^\infty \phi_\alpha(u\langle y,x\rangle)u^{-\alpha-1}e^{-h^{-1/\alpha}u}du\right| &\le \int_0^\infty |e^{iu\langle y,x\rangle} - 1 - iu\langle y,x\rangle|u^{-\alpha-1}du \\
&\le \int_0^\infty \left(\frac{1}{2}(u\langle y,x\rangle)^2 \wedge 2|u\langle y,x\rangle|\right)u^{-\alpha-1}du \\
&= \|y\|^\alpha\|x\|^\alpha \int_0^\infty \left(\frac{1}{2}s^2 \wedge 2s\right)s^{-\alpha-1}ds,
\end{aligned}$$

where $\int_0^\infty (s^2/2 \wedge 2s)s^{-\alpha-1}ds < \infty$. Thus, the integrability condition (2.2.2) of ρ justifies the interchange for $\alpha \ne 1$. When $\alpha = 1$, we have $|\phi_1(h,s)| = C|s|(1 + |\ln(1 + |s|)|)$ for $h \le 1$, which concludes the proof.

On the other hand, in long period of time, tempered stable processes behave like Gaussian processes.

Theorem 2.3.5. *Let $\{X_t : t \ge 0\}$ be a tempered stable process in $\mathbb{R}^d$ with $TS(\alpha,\rho;0)$. Assume that*

$$\int_{\mathbb{R}^d_0} ||x||^2\rho(dx) < \infty. \tag{2.3.4}$$

If $\alpha \in [1,2)$, *then*

$$\{h^{-1/2}X_{ht} : t \geq 0\} \xrightarrow{d} \{B_t : t \geq 0\} \quad as \quad h \to \infty,$$

where $\{B_t : t \geq 0\}$ *is a Brownian motion with characteristic function*

$$E[e^{i\langle y, B_t\rangle}] = \exp\left[-\frac{t}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0^d}\langle y, x\rangle^2\rho(dx)\right].$$

If $\alpha \in (0,1)$ *and* $\int_{\mathbb{R}_0^d}||x||\rho(dx) < \infty$, *then*

$$\{h^{-1/2}(X_{ht} - b\,ht) : t \geq 0\} \xrightarrow{d} \{B_t : t \geq 0\} \quad as \quad h \to \infty,$$

where $\{B_t : t \geq 0\}$ *is as above and* $b = \Gamma(1-\alpha)\int_{\mathbb{R}_0^d} x\rho(dx)$.

Proof. As in the preceding theorem, we will only consider the marginal at time 1. Observe that $E[e^{i\langle y, h^{-1/2}X_h\rangle}]$ for $\alpha \in [1,2)$ and $E[e^{i\langle y, h^{-1/2}(X_h - hb)\rangle}]$ for $\alpha \in (0,1)$ have the characteristic function

$$\exp\left[\int_{\mathbb{R}_0^d} h\phi_\alpha(h^{-1/2}\langle y, x\rangle)\rho(dx)\right],$$

where

$$\phi_\alpha(u) = \int_0^\infty (e^{ius} - 1 - ius)s^{-\alpha-1}e^{-s}ds.$$

There exists $\theta \in \mathbb{C}$ satisfying $|\theta| \leq 1$ and

$$\phi_\alpha(u) = \int_0^\infty \left(\frac{(ius)^2}{2!} + \theta\frac{(ius)^3}{3!}\right)s^{-\alpha-1}e^{-s}ds.$$

Therefore, we get

$$\lim_{h\to\infty} h\phi_\alpha(h^{-1/2}s\langle y, x\rangle) = -\frac{1}{2}\Gamma(2-\alpha)\langle y, x\rangle^2.$$

Finally,

$$|\phi_\alpha(u)| \leq \int_0^\infty |e^{-ius} - 1 - ius|s^{-\alpha-1}ds \leq u^2\Gamma(2-\alpha)$$

with (2.3.4) justifies the interchange of the limit with the integral. The proof is complete.

We have seen in Proposition 2.2.8 that a tempered stable process has exponential moment if the inner measure ρ has compact support. Under such an assumption, a generalized Berry-Essen theorem asserts that the long-time Gaussian convergence should be extremely fast. (Recall that the Gaussian convergence of truncated Lévy flights (TLF) is very slow.) Figure 2.2 compares tails of tempered stable density, its short-time α-stable density, and its long-time Gaussian density.

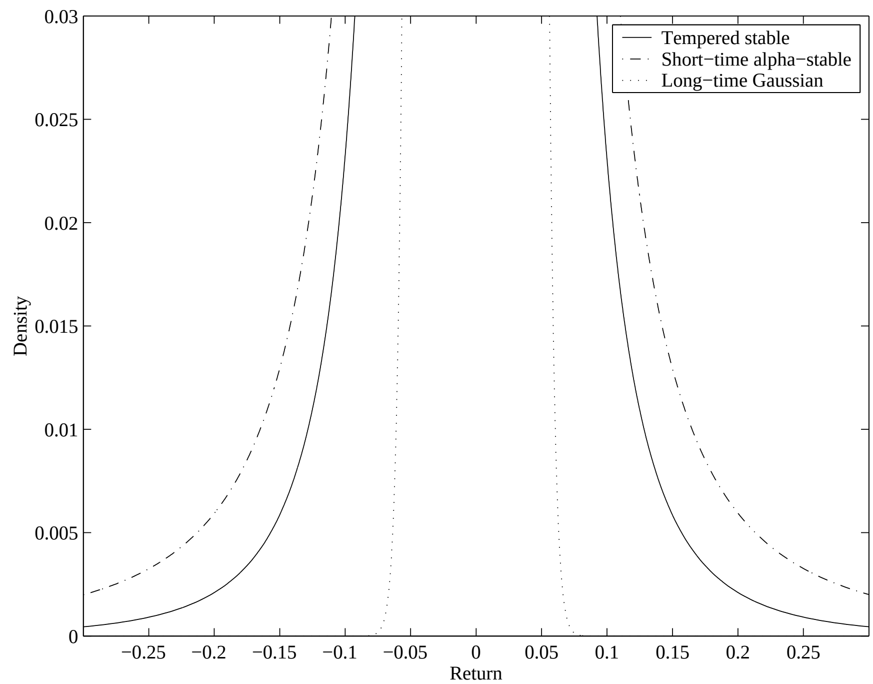


Figure 2.2: Comparison of tail behaviors

The tempered stable tails decay much slower than the Gaussian tails of e^{-x^2} order, and faster than the α-stable tails of $x^{-\alpha}$ order. Interestingly enough, the tempered stable law is more similar to the α-stable in terms of density tails, while its moment property is closer to Gaussian.

We have seen that a tempered stable process looks like a stable process in a local sense. Indeed, there is a deeper relationship between their laws. Under some suitable conditions, the law of tempered stable processes are absolutely continuous with respect to that of stable processes. The following is a direct consequence of

Theorem 33.1 of Sato [44]. Recall that $\mathbb{D}$ denotes the space of processes with cádlág paths.

Proposition 2.3.6. *Let ν and ν_α be given by (2.2.5) and (2.2.7), respectively. Let $\{X_t : t \in [0,T]\}$ be a tempered α-stable process in $\mathbb{R}^d$ generated by $(\gamma_1, 0, \nu)$, and $\{Y_t : t \in [0,T]\}$ an α-stable process in $\mathbb{R}^d$ generated by $(\gamma_2, 0, \nu_\alpha)$. If*

$$\int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \left(\sqrt{q(r,\xi)} - \sqrt{q(0+,\xi)}\right)^2 \frac{dr}{r^{\alpha+1}} < \infty \tag{2.3.5}$$

and

$$\gamma_2 - \gamma_1 - \int_{S^{d-1}} \sigma(d\xi) \int_0^1 (q(r,\xi) - q(0+,\xi)) \frac{dr}{r^\alpha} = 0,$$

then the law of $\{X_t\}$ and that of $\{Y_t\}$ in $\mathbb{D}[0,T]$ are mutually absolutely continuous.

Proof. Clear by Theorem 33.1 of Sato [44] with the following Radon-Nykodym derivative;

$$\frac{d\nu}{d\nu_\alpha}(x) = \begin{cases} \frac{q(\|x\|, x/\|x\|)}{q(0+, x/\|x\|)}, & \text{if } q(0+, x/\|x\|) > 0, \\ 1, & \text{if } q(0+, x/\|x\|) = 0. \end{cases}$$

The following is a direct consequence of Theorem 33.2 of Sato [44] and the preceding proposition.

Corollary 2.3.7. *Suppose that* (2.3.5) *of Proposition 2.3.6 holds and that $\nu, \nu_\alpha, \gamma_1$, and γ_2 be defined as in Proposition 2.3.6. Let $\{Y_t : t \geq 0\}$ be an α-stable process generated by $(\gamma_1, 0, \nu_\alpha)$ on a probability space $(\Omega, \mathcal{F}, P)$, admitting the Lévy-Ito form*

$$Y_t = \gamma_1 t + \int_0^t \int_{\|z\|<1} z(\mu_\alpha - \nu_\alpha)(dz, ds) + \int_0^t \int_{\|z\|\geq 1} z\mu_\alpha(dz, ds), \quad t \geq 0,$$

where μ_α is a Poisson random measure with intensity measure ν_α. Let $\{U_t : t \geq 0\}$ be a stochastic process in $\mathbb{R}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ by

$$U_t := \lim_{\epsilon \downarrow 0} \left(\int_0^t \int_{\|z\|>\epsilon} \ln\left(\frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)}\right) \mu_\alpha(dz, ds) \right.$$
$$\left. + t \int_{S^{d-1}} \sigma(d\xi) \int_\epsilon^\infty (q(0+,\xi) - q(r,\xi)) \frac{dr}{r^{\alpha+1}} \right), \quad t \geq 0,$$

where the convergence is uniform in t on any bounded interval. Define a probability measure Q by

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = e^{U_t}, \quad P\text{−}a.s.$$

Then, $\{Y_t : t \geq 0\}$ is a tempered α-stable process generated by $(\gamma_2, 0, \nu)$ with respect to the probability measure Q.

By a similar argument, we can derive a mutual absolute continuity within the class of tempered stable processes. Let

$$\nu_i(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) q_i(r,\xi) \frac{dr}{r^{\alpha+1}}, \quad i = 1, 2,$$

where q_1, q_2 and σ satisfy all conditions so that ν_1 and ν_2 are Lévy measures of tempered stable distributions, i.e., for each i, $q_i(\cdot, \xi)$ is completely monotone with $\lim_{r\to\infty} q_i(r,\xi) = 0$ for every $\xi \in S^{d-1}$, and σ is a probability measure on S^{d-1} such that $\int_{S^{d-1}} q_i(0+, \xi)\sigma(d\xi) < \infty$. Moreover, for each i, let $\{X_t^i : t \geq 0\}$ be a tempered α-stable process in $\mathbb{R}^d$ on probability spaces $(\Omega^i, \mathcal{F}^i, P^i)$ generated by $(\gamma_i, 0, \nu_i)$, admitting the Lévy-Itô form,

$$X_t^i = \gamma_i t + \int_0^t \int_{\|z\|\leq 1} z(\mu_i - \nu_i)(dz, ds) + \int_0^t \int_{\|z\|>1} z\mu_i(dz, ds), \quad t \geq 0,$$

where μ_i are Poisson random measures on $(\Omega^i, \mathcal{F}^i, P^i)$ with intensity measure ν^i, respectively.

Proposition 2.3.8. *If*

$$\int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \left(\sqrt{q_1(r,\xi)} - \sqrt{q_2(r,\xi)}\right)^2 \frac{dr}{r^{\alpha+1}} < \infty$$

and

$$\gamma_2 - \gamma_1 - \int_{\|z\|\leq 1} z(\nu_2 - \nu_1)(dz) = 0,$$

then for every $t \in (0,\infty)$, $P^1|_{\mathcal{F}_t^1}$ and $P^2|_{\mathcal{F}_t^2}$ are equivalent probability measures. Moreover, define

$$\begin{aligned} U_t^1 &:= \lim_{\epsilon \downarrow 0} \Big(\int_0^t \int_{\|z\|>\epsilon} \ln \left(\frac{q_1(\|z\|, z/\|z\|)}{q_2(\|z\|, z/\|z\|)} \right) \mu_\alpha^1(dz, ds) \\ &\qquad -t \int_{S^{d-1}} \sigma(d\xi) \int_\epsilon^\infty (q_2(r,\xi) - q_1(r,\xi)) \frac{dr}{r^{\alpha+1}} \Big), \quad P^1\text{−}a.s. \end{aligned}$$

and

$$\begin{aligned} U_t^2 &:= \lim_{\epsilon \downarrow 0} \Big(\int_0^t \int_{\|z\|>\epsilon} \ln \left(\frac{q_2(\|z\|, z/\|z\|)}{q_1(\|z\|, z/\|z\|)} \right) \mu_\alpha^2(dz, ds) \\ &\qquad -t \int_{S^{d-1}} \sigma(d\xi) \int_\epsilon^\infty (q_1(r,\xi) - q_2(r,\xi)) \frac{dr}{r^{\alpha+1}} \Big), \quad P^2\text{−}a.s. \end{aligned}$$

Then, we have

$$\frac{dP^2}{dP^1}|_{\mathcal{F}_t^1} = e^{U_t^1}, \quad P^1\text{−}a.s. \quad and \quad \frac{dP^1}{dP^2}|_{\mathcal{F}_t^2} = e^{U_t^2}, \quad P^2\text{−}a.s.$$

In particular, $\{X_t^1 : t \geq 0\}$ is a tempered α-stable process generated by $(\gamma_2, 0, \nu_2)$ with respect to the probability measure P^2, and similarly, $\{X_t^2 : t \geq 0\}$ is a tempered α-stable process generated by $(\gamma_1, 0, \nu_1)$ with respect to the probability measure P^1.

We close this section with discussion of the Gaussian approximation of the small jump component. We only need to check the conditions in Proposition 1.4.5.

Proposition 2.3.9. *Let $\{X_t : t \geq 0\}$ be a tempered stable process with $TS(\alpha, \rho; 0)$ in $\mathbb{R}^d$ and $\{B_t : t \geq 0\}$ be a d-dimensional standard Brownian motion. Define*

$$X_t^\epsilon := \sum_{s \leq t} \Delta X_s 1(\|\Delta X_s\| \leq \epsilon) - E\left[\sum_{s \leq t} \Delta X_s 1(\|\Delta X_s\| \leq \epsilon) \right]$$

and $\sigma^2(\epsilon) := \int_{\mathbb{R}_0^d} xx^T \int_{\epsilon/\|x\|}^\infty s^{1-\alpha} e^{-s} ds \rho(dx)$. Then,

$$\{\sigma(\epsilon)^{-1} X_t^\epsilon : t \geq 0\} \xrightarrow{d} \{B_t : t \geq 0\} \quad as \quad \epsilon \to 0.$$

Proof. By Proposition 1.4.5, it suffices to show that $\|\epsilon^2 \sigma^2(\epsilon)^{-1}\|_O \to 0$ as $\epsilon \to 0$. We have

$$\sigma^2(\epsilon) = (2-\alpha)^{-1} \int_{\mathbb{R}_0^d} xx^T \left(e^{-\epsilon/\|x\|} \left(\frac{\epsilon}{\|x\|} \right)^{2-\alpha} + \int_0^{\epsilon/\|x\|} e^{-s} s^{2-\alpha} ds \right) \rho(dx)$$

and hence for some $C > 0$,

$$\|\epsilon^2\sigma^2(\epsilon)^{-1}\|_O \leq C\epsilon^\alpha(2-\alpha)\left(\int_{\mathbb{R}_0^d} e^{-\epsilon/\|x\|}\|x\|^\alpha \rho(dx)\right)^{-1}.$$

Clearly, the right hand side tends to 0 as $\epsilon \to 0$. The proof is complete.

2.4 Series Representations

In this section, we derive two forms of series representation of tempered stable processes, one via the generalized shot noise method and the other via the rejection method. (Recall Theorem 1.4.1 and the statements thereafter.) The one via the generalized shot noise method is very useful for a simulation of sample path of tempered stable processes. On the other hand, the one via the rejection method will be used for studying the tail behaviors of tempered stable distributions.

Proposition 2.4.1. *Let $\{X_t : t \in [0,T]\} \sim TS(\alpha, \rho; 0)$ in $\mathbb{R}^d$. Then,*

$$\begin{aligned}\{X_t &: t \in [0,T]\}\\ &\stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}\|V_i\| \right) \frac{V_i}{\|V_i\|} 1(T_i \leq t) \right.\right.\\ &\qquad\qquad \left.\left. - m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha} k\, \frac{t}{T} \right] + b_T\, t : t \in [0,T] \right\},\end{aligned}$$

where the equality holds for finite dimensional distributions and the convergence on the right hand side holds a.s uniformly on $t \in [0,T]$. Here, $\{T_i\}_{i\geq 1}$ are i.i.d. uniform in $[0,T]$, $\{U_i\}_{i\geq 1}$ are iid uniform in $[0,1]$, $\{\Gamma_i\}_{i\geq 1}$ are arrival times of a standard Poisson process, $\{E_i\}_{i\geq 1}$ are i.i.d. exponential with parameter 1 with and $\{V_i\}_{i\geq 1}$ are iid random vectors in $\mathbb{R}_0^d$ with common distribution ρ_1 defined by

$$\rho_1(dx) = \frac{1}{m(\rho)^\alpha}||x||^\alpha \rho(dx)$$

where $m(\rho)^\alpha = \int_{\mathbb{R}_0^d} ||x||^\alpha \rho(dx)$. Moreover, $\{T_i\}_{i\geq 1}$, $\{\Gamma_i\}_{i\geq 1}$, $\{E_i\}_{i\geq 1}$, and $\{V_i\}_{i\geq 1}$ are *mutually independent. Finally, k and b_T are constants given by*

$$k = \begin{cases} 0, & if \ \alpha \in (0,1), \\ m(\rho)^{-\alpha} \int_{\mathbb{R}_0^d} x\|x\|^{\alpha-1}\rho(dx), & if \ \alpha \in [1,2), \end{cases}$$

and

$$b_T = \begin{cases} 0, & if \ \alpha \in (0,1), \\ T\left((\ln(Tm(\rho)) + 2\gamma) \int_{\mathbb{R}_0^d} x\rho(dx) - \int_{\mathbb{R}_0^d} x \ln\|x\|\rho(dx) \right), & if \ \alpha = 1, \\ k\, m(\rho)T^{-1}(\alpha/T)^{-1/\alpha}\zeta(-1/\alpha) + |\Gamma(1-\alpha)| \int_{\mathbb{R}_0^d} x\rho(dx), & if \ \alpha \in (1,2), \end{cases}$$

where ζ is the Riemann zeta function and γ is the Euler constant.

Proof. In view of Theorem 1.4.1, we only need to find a H-sequence of the Lévy measure of a tempered stable distribution, and its centering constants. Now, for each $b > 0$ and $C \in \mathcal{B}(S^{d-1})$, the Lévy measure ν of a tempered stable distribution can be decomposed as follows,

$$\begin{aligned}
\nu((b,\infty)C) &= \int_{\mathbb{R}_0^d}\int_0^\infty 1_{(b,\infty)C}(s\|x\| \cdot x/\|x\|)s^{-\alpha-1}e^{-s}ds\,\rho(dx) \\
&= \int_{\mathbb{R}_0^d}\int_0^\infty 1_{(b,\infty)}(s\|x\|)s^{-\alpha-1}e^{-s}ds\,1_C(x/\|x\|)\rho(dx) \\
&= \int_{\mathbb{R}_0^d} \alpha^{-1}\Big[(b/\|x\|)^{-\alpha}e^{-b/\|x\|} \\
&\qquad - \int_0^\infty 1_{(b,\infty)}(s\|x\|)s^{-\alpha}e^{-s}ds\Big]1_C(x/\|x\|)\rho(dx) \\
&= \int_{\mathbb{R}_0^d}\int_0^\infty 1_{(b,\infty)}(s\|x\|)\frac{b^{-\alpha}-(s\|x\|)^{-\alpha}}{\alpha}e^{-s}ds1_C(x/\|x\|)\|x\|^\alpha\rho(dx) \\
&= \int_{\mathbb{R}_0^d}\int_0^\infty\int_0^1 1_{(b,\infty)}(su^{1/\alpha}\|x\|)\frac{1}{\alpha}\left(\frac{m(\rho)}{b}\right)^{-\alpha}du\,e^{-s}ds1_C(x/\|x\|)\rho_1(dx) \\
&= \int_{\mathbb{R}_0^d}\int_0^\infty\int_0^1\int_0^\infty 1_{(b,\infty)}(m(\rho)(\alpha t)^{-1/\alpha} \wedge su^{1/\alpha}\|x\|) \\
&\qquad dt\,du\,e^{-s}ds1_C(x/\|x\|)\rho_1(dx) \\
&= \int_{\mathbb{R}_0^d}\int_0^\infty\int_0^1\int_0^\infty 1_{(b,\infty)C}((m(\rho)(\alpha t)^{-1/\alpha} \wedge su^{1/\alpha}\|x\|)\cdot x/\|x\|) \\
&\qquad dt\,du\,e^{-s}ds\rho_1(dx).
\end{aligned}$$

The fifth and sixth equalities hold since for each $a > 0$, $b > 0$, and positive invertible f,

$$\int_0^1 1_{(b,\infty)}(af(x))dx = \int_0^1 1_{(b/a,\infty)}(f(x))dx = Leb((0,1) \cap f^{-1}(b/a,\infty)),$$

and

$$\int_0^\infty 1_{(b,\infty)}(a \wedge f(x))dx = 1_{(b,\infty)}(a) \int_0^\infty 1_{(b,\infty)}(f(x))dx = 1_{(b,\infty)}(a) Leb(f^{-1}(b,\infty)),$$

respectively. Hence, an H-sequence is given by

$$H_i := \left(m(\rho)(\alpha\Gamma_i)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}\|V_i\|\right) \cdot \frac{V_i}{\|V_i\|}.$$

From the construction of the notation $TS(\alpha,\rho;0)$, the centering constants are given by

$$c_i(T) = \begin{cases} 0, & \text{if } \alpha \in (0,1), \\ \int_{i-1}^i E\left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}\|V_i\|\right)\frac{V_i}{\|V_i\|}\right] ds, & \text{if } \alpha \in [1,2). \end{cases}$$

Hence, the case $\alpha \in (0,1)$ is proved. For the case $\alpha \in [1,2)$, it remains to show that

$$\sum_{i=1}^\infty \left[m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha} k\frac{1}{T} - c_i\right] = b_T.$$

Set

$$d_i = \int_{i-1}^i E\left[m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha} \frac{V_i}{\|V_i\|}\right] ds,$$

for $i \geq 1$ when $\alpha \in (1,2)$, and for $i \geq 2$ when $\alpha = 1$ with $d_1 = 0$. Then, it is enough to show two approximations;

$$\sum_{i=1}^\infty \left(m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha} k - d_i\right) = \begin{cases} m(\rho)T\gamma\, k, & \text{if } \alpha = 1, \\ m(\rho)\,(\alpha/T)^{-1/\alpha}\,\zeta(-1/\alpha)\,k, & \text{if } \alpha \in (1,2), \end{cases} \tag{2.4.1}$$

and

$$\sum_{i=1}^\infty (d_i - c_i) = \begin{cases} T\left[(\ln(m(\rho)T) + \gamma)\int_{\mathbb{R}_0^d} x\rho(dx) - \int_{\mathbb{R}_0^d} x \ln\|x\|\rho(dx)\right], & \text{if } \alpha = 1, \\ T|\Gamma(1-\alpha)|\int_{\mathbb{R}_0^d} x\rho(dx) k, & \text{if } \alpha \in (1,2). \end{cases}$$

The first approximation holds as follows. For $\alpha = 1$, by the definition of the Euler constant,

$$\sum_{i=1}^{n} \left(m(\rho) \left(\frac{i}{T} \right)^{-1} k - d_i \right) = m(\rho) T k \left(\sum_{i=1}^{n} i^{-1} - \int_1^n s^{-1} ds \right)$$
$$= m(\rho) T k \left(\sum_{i=1}^{n} i^{-1} - \ln n \right) \to m(\rho) T k \gamma,$$

as $n \to \infty$. For $\alpha \in (1, 2)$, we have

$$\sum_{i=1}^{n} \left(m(\rho) \left(\frac{\alpha\, i}{T} \right)^{-1/\alpha} - \int_{i-1}^{i} m(\rho) \left(\frac{\alpha\, s}{T} \right)^{-1/\alpha} ds \right)$$
$$= m(\rho) \left(\frac{\alpha}{T} \right)^{-1/\alpha} \left(\sum_{i=1}^{n} i^{-1/\alpha} - \frac{n^{1-1/\alpha}}{1 - 1/\alpha} \right)$$
$$= m(\rho) \left(\frac{\alpha}{T} \right)^{-1/\alpha} \left(\zeta(-1/\alpha) + (1 - 1/\alpha) \int_n^\infty \frac{s - [s]}{s^{2-1/\alpha}} ds \right)$$
$$\to m(\rho) \left(\frac{\alpha}{T} \right)^{-1/\alpha} \zeta(-1/\alpha),$$

as $n \to \infty$.

Now, we will prove the second approximation. For $\alpha = 1$, first observe that for each $\theta > 0$,

$$\int_1^\infty \left(m(\rho) \left(\frac{s}{T} \right)^{-1} - m(\rho) \left(\frac{s}{T} \right)^{-1} \wedge \theta \right) ds$$
$$= 1(\theta \le m(\rho) T) \left(m(\rho) T \ln \left(\frac{m(\rho) T}{\theta} \right) - m(\rho) T + \theta \right)$$
$$\le m(\rho) T \ln^+ \left(\frac{m(\rho) T}{\theta} \right),$$

$$\int_0^1 \left(m(\rho) \left(\frac{s}{T} \right)^{-1} \wedge \theta \right) ds$$
$$= 1(\theta \le m(\rho) T) \theta + 1(\theta > m(\rho) T) m(\rho) T \left(1 - \ln \left(\frac{m(\rho) T}{\theta} \right) \right)$$
$$\le m(\rho) T (2 + \ln^+ \left(\frac{\theta}{m(\rho) T} \right),$$

and thus

$$-\int_0^1 \left(m(\rho) \left(\frac{s}{T} \right)^{-1} \wedge \theta \right) ds + \int_1^\infty \left(m(\rho) \left(\frac{s}{T} \right)^{-1} - m(\rho) \left(\frac{s}{T} \right)^{-1} \wedge \theta \right) ds$$
$$= m(\rho) T \left(\ln \left(\frac{m(\rho) T}{\theta} \right) - 1 \right).$$

Next, we have

$$\sum_{i=1}^{\infty} \|d_i - c_i\| = \|c_1\| + \sum_{i=2}^{\infty} \|d_i - c_i\|$$
$$\leq E\Bigg[\int_0^1 \left(m(\rho) \left(\frac{s}{T}\right) \wedge E_1 U_1 \|V_1\| \right) ds$$
$$+ \int_1^{\infty} \left(m(\rho) \left(\frac{s}{T}\right) - m(\rho) \left(\frac{s}{T}\right) \wedge E_1 U_1 \|V_1\| \right) ds \Bigg]$$
$$\leq m(\rho)T \left(2 + 2|\ln(m(\rho)T)| + E[|\ln(E_1U_1)|] + \frac{\int_{\mathbb{R}_0^d} |\ln\|x\|| \|x\| \rho(dx)}{m(\rho)} \right).$$

The probability measure of the random variable $E_1 U_1$ is

$$P(E_1U_1 \in (0,z)) = \int_0^1 \int_0^{\infty} 1_{(0,z)}(xy) e^{-x} dx dy = \int_1^{\infty} (1 - e^{-zs}) s^{-2} ds.$$

Thus, its density is given by the exponential integral function $E_1(z) = \int_1^{\infty} s^{-1} e^{-zs} ds$. Using a known result $\gamma = -\int_0^{\infty} \ln z \, e^{-z} dz$, we get

$$E[\ln(E_1U_1)] = \int_0^{\infty} \ln z \, E_1(z) dz$$
$$= \int_1^{\infty} s^{-1} \left(\int_0^{\infty} \ln z \, e^{-z} dz \right) ds - \int_1^{\infty} \ln s \int_0^{\infty} e^{-z} dz ds = -\gamma - 1.$$

Thus, $E[|\ln(E_1U_1)|] < \infty$ also follows and we get

$$\begin{aligned}
\sum_{i=1}^{\infty} (d_i - c_i) &= E\Bigg[\Bigg(-\int_0^1 \left(m(\rho) \left(\frac{s}{T}\right)^{-1} \wedge E_1 U_1 \|V_1\| \right) ds \\
&\quad + \int_1^{\infty} \left(m(\rho) \left(\frac{s}{T}\right)^{-1} - m(\rho) \left(\frac{s}{T}\right)^{-1} \wedge E_1 U_1 \|V_1\| \right) ds \Bigg) \frac{V_1}{\|V_1\|} \Bigg] \\
&= m(\rho) T E \left[\left(\ln(m(\rho)T) - \ln(E_1 U_1 \|V_1\|) - 1 \right) \frac{V_1}{\|V_1\|} \right] \\
&= m(\rho) T E \left[\left(\ln(m(\rho)T) + \gamma - \ln\|V_1\| \right) \frac{V_1}{\|V_1\|} \right] \\
&= T \left((\ln(m(\rho)T) + \gamma) \int_{\mathbb{R}_0^d} x \rho(dx) - \int_{\mathbb{R}_0^d} x \ln\|x\| \rho(dx) \right).
\end{aligned}$$

Finally, we consider the case $\alpha \in (1,2)$. We have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \left\| \int_{i-1}^{i} m(\rho) \left(\frac{\alpha\, s}{T}\right)^{-1/\alpha} ds - c_i(T) \right\| \\
&\qquad \le E\left[\int_0^\infty \left(m(\rho) \left(\frac{\alpha\, s}{T}\right)^{-1/\alpha} - m(\rho) \left(\frac{\alpha\, s}{T}\right)^{-1/\alpha} \wedge E_1 U_1^{1/\alpha} \|V_1\| \right) ds \right] \\
&\qquad = \frac{m(\rho)^\alpha T}{\alpha(\alpha-1)} E[(E_1 U_1^{1/\alpha} \|V_1\|)^{1-\alpha}] \\
&\qquad = T|\Gamma(1-\alpha)| \int_{\mathbb{R}_0^d} \|x\| \rho(dx) < \infty,
\end{aligned}$$

where the first equality holds by the pointwise use of

$$\int_0^\infty \left(m(\rho) \left(\frac{\alpha s}{T}\right)^{-1/\alpha} - m(\rho) \left(\frac{\alpha s}{T}\right)^{-1/\alpha} \wedge \theta \right) ds = \frac{m(\rho)^\alpha T}{\alpha(\alpha-1)} \theta^{1-\alpha}, \quad \theta > 0.$$

Hence, we get

$$\begin{aligned}
&\sum_{i=1}^{\infty} (d_i - c_i) \\
&\qquad = E\left[\int_0^\infty \left(m(\rho) \left(\frac{\alpha\, s}{T}\right)^{-1/\alpha} - m(\rho) \left(\frac{\alpha\, s}{T}\right)^{-1/\alpha} \wedge E_1 U_1^{1/\alpha} \|V_1\| \right) ds \, \frac{V_1}{\|V_1\|} \right] \\
&\qquad = \frac{m(\rho)^\alpha T}{\alpha(\alpha-1)} E\left[E_1^{1-\alpha} U_1^{1/\alpha - 1} \frac{V_1}{\|V_1\|^\alpha} \right] \\
&\qquad = T|\Gamma(1-\alpha)| \int_0^d x \rho(dx),
\end{aligned}$$

which proves the case $\alpha \in (1,2)$. The proof is complete.

Clearly, the terms $\{E_i U_i^{1/\alpha} \|V_i\|\}_{i \ge 1}$ are exponential temperings. Thanks to the simple structure, this series representation can be used for simulation of tempered stable random vectors or processes. We give below typical sample paths of tempered stable processes generated by the series representation. For simplicity, the inner measure ρ is set $\rho(dx) = \delta_{-1.0}(dx) + \delta_{1.0}(dx)$.

In the next proposition, we will derive a different series representation via the rejection method.

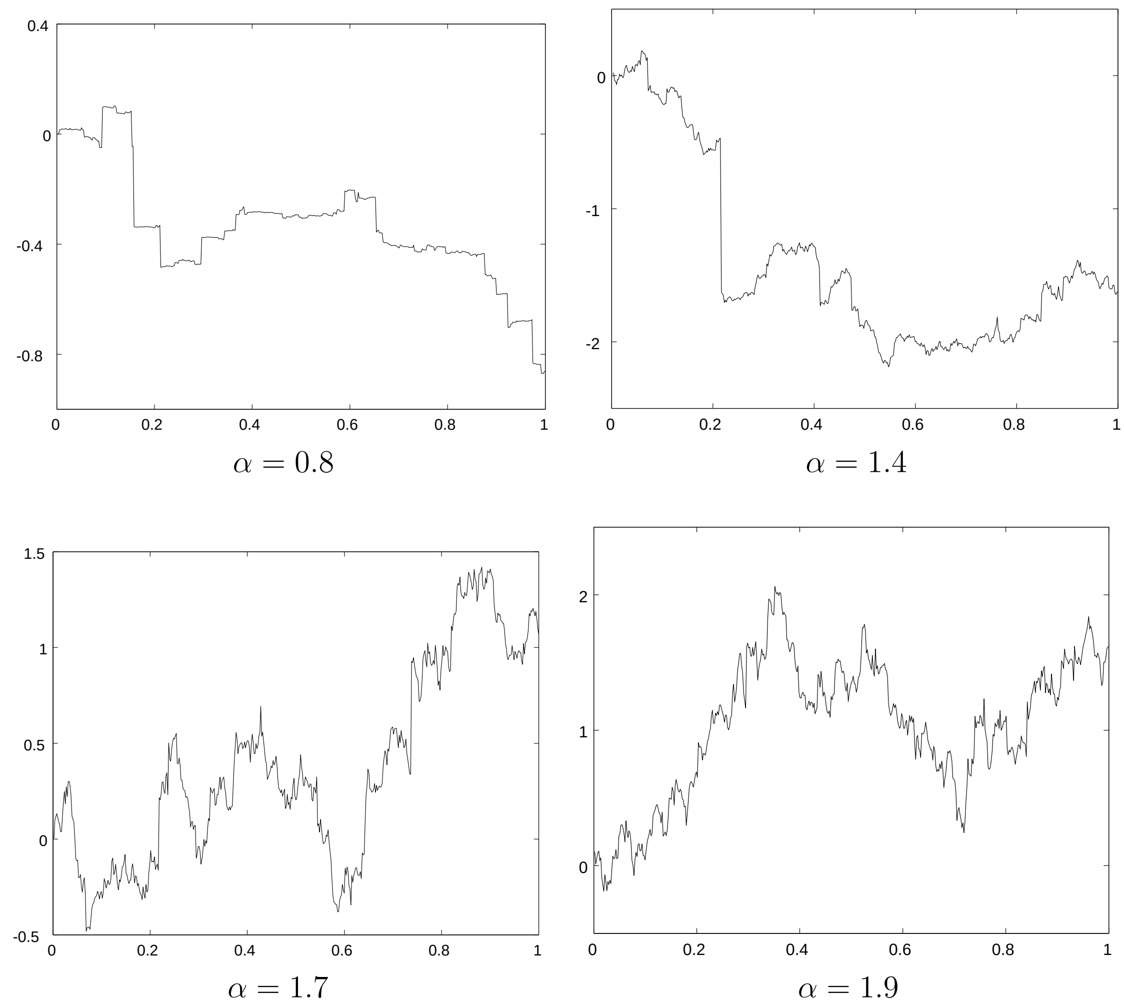


Figure 2.3: Tempered stable processes generated via the series representation

Proposition 2.4.2. *Let* $\{X_t : t \in [0,T]\} \sim TS(\alpha, \rho; 0)$ *in* $\mathbb{R}^d$*. Set*

$$h(z) = \begin{cases} \frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)}, & if\ q(0+, z/\|z\|) > 0 \\ 1, & otherwise. \end{cases}$$

Then,

$$\{X_t : t \in [0,T]\} \stackrel{d}{=} \Bigg\{ \sum_{i=1}^{\infty} \Bigg(J_i^1 \, 1(h(J_i^1) \geq U_i) \, 1(T_i \leq t) - \left[m(\rho) \left(\frac{\alpha i}{T} \right)^{-1/\alpha} - m(\rho)^2 \left(\frac{\alpha i}{T} \right)^{-2/\alpha} \right] k \frac{t}{T} \Bigg) + b_T' t : t \in [0,T] \Bigg\},$$

where the equality holds for finite dimensional distributions and the convergence on *the right hand side holds a.s uniformly in* $t \in [0,T]$*. Here,*

$$J_i^1 = m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\frac{V_i}{\|V_i\|}$$

where $\{U_i\}_{i\geq 1}$*,* $\{T_i\}_{i\geq 1}$*,* $\{\Gamma_i\}_{i\geq 1}$*,* $\{V_i\}_{i\geq 1}$*,* ρ_1*,* $m(\rho)$*,* k *are defined as in Theorem 2.4.1, and*

$$b'_T = \begin{cases} 0, & if \ \alpha \in (0,1), \\ k\, m(\rho)T(\gamma - m(\rho)T\zeta(-2)), & if \ \alpha = 1, \\ k\, m(\rho)\left(\frac{\alpha}{T}\right)^{-1/\alpha}\left(\zeta\left(-\frac{1}{\alpha}\right) - m(\rho)\left(\frac{\alpha}{T}\right)^{-1/\alpha}\zeta\left(-\frac{2}{\alpha}\right)\right), & if \ \alpha \in (1,2). \end{cases}$$

We prepare a lemma.

Lemma 2.4.3. *Let* $\{Y_t : t \in [0,T]\}$ *be an* α*-stable process given by (2.3.2)-(2.3.3). Then,*

$$\{Y_t : t \in [0,T]\} \stackrel{d}{=} \left\{\sum_{i=1}^{\infty}\left[J_i^1\, 1(T_i \leq t) - m(\rho)\left(\frac{\alpha i}{T}\right)^{-1/\alpha} k\frac{t}{T}\right] + b''_T\, t : t \in [0,T]\right\} \tag{2.4.2}$$

where random sequences $\{T_i\}_{i\geq 1}$*,* $\{\Gamma_i\}_{i\geq 1}$*,* $\{V_i\}_{i\geq 1}$*, and constants* k *and* $m(\rho)$ *are defined as in Theorem 2.4.1,* $\{J_i^1\}_{i\geq 1}$ *is defined as in Theorem 2.4.2, and*

$$b''_T = \begin{cases} 0, & if \ \alpha \in (0,1), \\ k\, m(\rho)T(2\gamma + \ln(m(\rho)T) - 1), & if \ \alpha = 1, \\ k\, m(\rho)(\alpha/T)^{-1/\alpha}\zeta(-1/\alpha), & if \ \alpha \in (1,2), \end{cases}$$

Proof. By Remark 2.2.4, we have

$$\nu_\alpha(B) = \int_{S^{d-1}}\sigma(d\xi)\int_0^\infty 1_B(r\xi)m(\rho)^\alpha\frac{dr}{r^{\alpha+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d).$$

An H-sequence is given by the LePage's method with the inverse q-function,

$$q^{\leftarrow}(s) := \inf\left\{x > 0 : \lambda(S^{d-1})\int_x^\infty r^{-\alpha-1}dr < s\right\} = m(\rho)(\alpha s)^{-1/\alpha}. \tag{2.4.3}$$

Now, we will consider the centering constants. For $\alpha \in (0,1)$, no centering terms are needed and thus the result follows. For $\alpha \in (1,2)$, it is enough to show

$$\sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{\alpha i}{T}\right)^{-1/\alpha}k - \int_{i-1}^{i} E\left[m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\frac{V_1}{\|V_1\|}\right]ds\right) = b_T'',$$

but this was indeed proved in Proposition 2.4.1 (See (2.4.1)). We will consider the case $\alpha = 1$. Observe that

$$\exp\left[T\int_{\mathbb{R}_0^d}\left(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle 1_{\{\|z\|\le 1\}}(z)\right)\nu_\alpha(dz) - i\langle y,(1-\gamma)m(\rho)Tk\rangle\right] = \widehat{\mu}_{Y_T}(y).$$

(See Exercise 18.15 of Sato [44].) Then, we get the result because

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(m(\rho)\left(\frac{i}{T}\right)^{-1}k - \int_{i-1}^{i} E\left[m(\rho)\left(\frac{s}{T}\right)^{-1}1\left(m(\rho)\left(\frac{s}{T}\right)^{-1}\le 1\right)\frac{V_1}{\|V_1\|}\right]ds\right)\\
&\qquad = m(\rho)T\sum_{i=1}^{\infty}\left(i^{-1} - \int_{(i-1)\vee m(\rho)T}^{i\vee m(\rho)T} s^{-1}ds\right)k\\
&\qquad = m(\rho)T\left(\lim_{n\to\infty}\left(\sum_{i=1}^{n} i^{-1} - \ln(n\vee m(\rho)T)\right) + \ln(m(\rho)T)\right)k\\
&\qquad = m(\rho)T(\gamma + \ln(m(\rho)T))k.
\end{aligned}$$

The proof is complete.

Proof of Theorem 2.4.2. By Proposition 2.2.3, we know that

$$\frac{d\nu}{d\nu_\alpha}(z) = \frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)} = g(z) \le 1 \tag{2.4.4}$$

where ν_α is the Lévy measure of Y_1. Hence, the rejection method gives a H-sequence

$$J_j = \begin{cases} J_j^1, & \text{if } g(J_j^1) \ge U_j \\ 0, & \text{otherwise.}\end{cases}$$

Now we will derive the centering constants for $\alpha \in [1,2)$. It is enough to show

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left(\left[m(\rho)\left(\frac{\alpha i}{T}\right)^{-1/\alpha} - m(\rho)^2\left(\frac{\alpha i}{T}\right)^{-2/\alpha}\right]\right.\\
&\qquad \left. - \int_{i-1}^{i} E\left[m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\frac{V_1}{\|V_1\|}1\left(m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\ge U_1\right)\right]ds\right) = b_T'.
\end{aligned}$$

Observing that for each $i \geq 1$,

$$\begin{aligned}\int_{i-1}^{i} E\left[m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\frac{V_1}{\|V_1\|}1\left(m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\geq U_1\right)\right]ds \\ = \int_{i-1}^{i} m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\left(1-m(\rho)\left(\frac{\alpha s}{T}\right)^{-1/\alpha}\right)ds\, k \\ = m(\rho)\left(\frac{\alpha}{T}\right)^{-1/\alpha}\int_{i-1}^{i} s^{-1/\alpha}ds\, k - m(\rho)^2\left(\frac{\alpha}{T}\right)^{-2/\alpha}\int_{i-1}^{i} s^{-2/\alpha}ds\, k,\end{aligned}$$

and

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} i^{-a} - \int_0^n s^{-a}ds\right) = \begin{cases}\zeta(-a), & \text{if } a>0,\ a\neq 1,\\ \gamma, & \text{if } a=1,\end{cases}$$

we get the result.

2.5 Tail Behaviors

In this section, we will discuss the probability tails of tempered stable distributions. Although their exact asymptotics are hard to obtain unlike those of stable distributions, the upper and lower bounds will give some nice intuition.

In the following, we provide an upper bound of the right probability tail following Theorem 5.2 of Breton et al.[8] Here, $m(X)$ denotes the median of the random variable X.

Proposition 2.5.1. *Let X be a random variable where $\mathcal{L}(X) \sim TS(\alpha,\rho;\gamma)$ and set*

$$\Theta(y) := y^{-\alpha}\int_{\mathbb{R}_0}|x|^{\alpha}\int_0^1 s^{-\alpha+1}e^{-\frac{y}{|x|}s}ds\rho(dx).$$

Then,

$$P(X - m(X) \geq \lambda) \leq (1+e)\Theta(\lambda/4)$$

for $\lambda \geq 2\Theta^{-1}(1/(2(1+e)))$.

Proof. Let ν be a Lévy measure of X, and let ν_y be a restriction $[\nu]_{|z|\leq y}$ for $y>0$. Now, fix $y>0$. We can think of X as the value X_1 of a Lévy process $\{X_t : t\in[0,1]\}$

given by

$$X_t = at + \int_0^t \int_{|z| \le y} z(\mu - \nu)(dz, ds) + \int_0^t \int_{|z|>y} z\mu(dz, ds),$$

where μ is a Poisson random measure with intensity measure ν and a is a suitable constant. Then, for $|z| \le y$ and $t \in [0,1]$, the Malliavin derivative of $\{X_t : t \in [0,1]\}$ is given by $D_{z,t}X_1 = z$ and thus $\sup_{|z|\le y} |D_{z,1}X_1| = y$. Moreover, we have

$$\begin{aligned}
\int_{\mathbb{R}_0} |D_{z,1}X_1|^2 [\nu]_{\|z\| \le y}(dz) &= \int_{\mathbb{R}_0} |x|^2 \int_0^{y/|x|} s^{-\alpha+1} e^{-s} ds \rho(dx) \\
&= \left(\sup_{|z| \le y} |D_{z,1}X_1| \right)^2 \Theta(y).
\end{aligned}$$

Then, a direct application of Theorem 5.2 of Breton et al.[8] gives the estimate.

For each $y > 0$, $\Theta(y)$ is bounded from above by $y^{-\alpha}(2-\alpha)^{-1} \int_{\mathbb{R}_0} |x|^\alpha \rho(dx)$, which leads to a less tight estimate

$$P(X - m(X) \ge y) \le y^{-\alpha} \frac{1+e}{4^\alpha (2-\alpha)} \int_{\mathbb{R}_0} |x|^\alpha \rho(dx),$$

for y large enough. This asserts, as Proposition 2.2.8 (i) implied, that the upper tail of tempered stable distributions is at most as heavy as that of stable distributions. Moreover, as a special case, if $\rho(dx) = 2^{-1}(\delta_{-1}(dx) + \delta_1(dx))$, then

$$\Theta(y) = y^{-2} \int_0^y s^{-\alpha+1} e^{-s} ds \sim \Gamma(2-\alpha) y^{-2},$$

as $y \to \infty$.

A natural further interest is in a lower bound of probability tails. We will below give a lower bound, following Lemma 5.4 of Breton et al.[8].

Proposition 2.5.2. *Let X be a random vector in $\mathbb{R}^d$ such that $\mathcal{L}(X) \sim TS(\alpha, \rho; \gamma)$. Then, for every $\lambda > 0$,*

$$P(\|X - m(X)\| \ge \lambda) \ge \frac{1}{4} \left(1 - \exp \left[- \int_{\mathbb{R}_0^d} \int_{2\lambda/\|x\|}^\infty s^{-\alpha-1} e^{-s} ds \rho(dx) \right] \right). \qquad (2.5.1)$$

In particular, if ρ is symmetric, then

$$P(\|X\| \geq \lambda) \geq \frac{1}{2}\left(1 - \exp\left[-\int_{\mathbb{R}_0^d}\int_{\lambda/\|x\|}^{\infty} s^{-\alpha-1}e^{-s}ds\rho(dx)\right]\right). \tag{2.5.2}$$

Proof. Let ν be the Lévy measure of X. Then, by Lemma 5.4 of Breton et al.[8], for each $\lambda > 0$,

$$P(\|X - m(X)\| \geq \lambda) \geq \frac{1}{4}\left(1 - \exp\left[-\nu\left(\{\|z\| \in \mathbb{R}_0^d : \|z\| \geq 2\lambda\}\right)\right]\right),$$

and when ν is symmetric,

$$P(\|X\| \geq \lambda) \geq \frac{1}{2}\left(1 - \exp\left[-\nu\left(\{\|z\| \in \mathbb{R}_0^d : \|z\| \geq \lambda\}\right)\right]\right).$$

For further analysis, we need the following asymptotics.

Lemma 2.5.3. *For $\alpha \in (0, 2)$,*

$$\int_\lambda^\infty s^{-\alpha-1}e^{-s}ds = \lambda^{-\alpha-1}e^{-\lambda} + o(\lambda^{-\alpha-1}e^{-\lambda}),$$

as $\lambda \to \infty$.

Proof. By integration by parts, we have

$$\int_\lambda^\infty s^{-\alpha-1}e^{-s}ds = \lambda^{-\alpha-1}e^{-\lambda} - (\alpha+1)\int_\lambda^\infty s^{-\alpha-2}e^{-s}ds \leq \lambda^{-\alpha-1}e^{-\lambda},$$

and

$$\begin{aligned}\int_\lambda^\infty s^{-\alpha-1}e^{-s}ds &= \lambda^{-\alpha-1}e^{-\lambda} - (\alpha+1)\lambda^{-\alpha-2}e^{-\lambda} + (\alpha+1)(\alpha+2)\int_\lambda^\infty s^{-\alpha-3}e^{-s}ds \\ &\geq \lambda^{-\alpha-1}e^{-\lambda} - (\alpha+1)\lambda^{-\alpha-2}e^{-\lambda},\end{aligned}$$

which gives the result.

Clearly, the exponents in the right hand sides of (2.5.1) and (2.5.2) tend to zero as $\lambda \to \infty$. Together with Lemma 2.5.3, the lower bounds above behave like, for the general case,

$$\frac{1}{4}\left(1 - \exp\left[-\int_{\mathbb{R}_0^d}\int_{2\lambda/\|x\|}^{\infty} s^{-\alpha-1}e^{-s}ds\rho(dx)\right]\right) \sim \frac{\lambda^{-\alpha-1}}{2^{\alpha+3}}\int_{\mathbb{R}_0^d}\|x\|^{\alpha+1}e^{-2\lambda/\|x\|}\rho(dx),$$

and for the symmetric case,

$$\frac{1}{2}\left(1 - \exp\left[-\int_{\mathbb{R}_0^d}\int_{\lambda/\|x\|}^{\infty} s^{-\alpha-1}e^{-s}ds\rho(dx)\right]\right) \sim \frac{\lambda^{-\alpha-1}}{2}\int_{\mathbb{R}_0^d}\|x\|^{\alpha+1}e^{-\lambda/\|x\|}\rho(dx),$$

as $\lambda \to \infty$. Moreover, as discussed above, if $\rho(dx) = 2^{-1}(\delta_{-1}(dx) + \delta_1(dx))$, then Thus, the lower bounds behave like, for the general case, $2^{-\alpha-3}\lambda^{-\alpha-1}e^{-2\lambda}$, and for the symmetric case, $2^{-1}\lambda^{-\alpha-1}e^{-\lambda}$.

The series representation obtained in Proposition 2.4.2 also provides a nice intuition of the tail behaviors of tempered stable distributions. Let X be a random vector in $\mathbb{R}^d$ where $\mathcal{L}(X) \sim TS(\alpha, \rho; 0)$ and the inner measure ρ is the uniform probability measure defined on S^{d-1}, i.e., the Lévy measure is given by

$$\nu(BC) = \int_0^{\infty} 1_B(r)\frac{e^{-r}}{r^{\alpha+1}}dr\,\rho(C), \quad B \in \mathcal{B}(0,\infty), \quad C \in \mathcal{B}(S^{d-1}). \tag{2.5.3}$$

Now, the structure (2.5.3) implies that

$$\frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)} = e^{-\|z\|}, \quad z \in \mathbb{R}_0^d,$$

and hence X admits a series representation

$$X \stackrel{d}{=} \sum_{i=1}^{\infty}(\alpha\Gamma_i)^{-1/\alpha}V_i 1((\alpha\Gamma_i)^{-1/\alpha} \leq E_i),$$

where $\{V_i\}_{i\geq 1}$ is a sequence of iid uniform vectors in S^{d-1}, $\{\Gamma_i\}_{i\geq 1}$ and $\{E_i\}_{i\geq 1}$ are defined as in Proposition 2.4.1. Set for each $i \geq 1$, $Y_i := V_i(\alpha\Gamma_i)^{-1/\alpha}1((\alpha\Gamma_i)^{-1/\alpha} \leq E_i)$. Then, we have the following.

Lemma 2.5.4. *Let $\{Y_i\}_{i\geq 1}$ be a sequence of random vectors in $\mathbb{R}^d$ defined as above. Then,*

$$P(\|Y_1\| > \lambda) \sim \alpha^{-1}\lambda^{-\alpha}e^{-\lambda}, \quad \textit{as } \lambda \to \infty.$$

Moreover,

$$E[\|Y_i\|^\alpha e^{\|Y_i\|}] = \begin{cases} \infty, & i = 1 \\ (\alpha(i-1))^{-1}, & i \geq 2. \end{cases}$$

Proof. Clearly, $\|Y_i\| = (\alpha\Gamma_i)^{-1/\alpha}1((\alpha\Gamma_i)^{-1/\alpha} \leq E_i)$. For the first claim, we have

$$\begin{aligned} P(\|Y_1\| > \lambda) &= P\left(\|V_1(\alpha\Gamma_1)^{-1/\alpha}1((\alpha\Gamma_1)^{-1/\alpha} \leq E_1)\| > \lambda\right) \\ &= P(\lambda < (\alpha\Gamma_1)^{-1/\alpha} \leq E_1) \\ &= \int_\lambda^\infty P(\lambda < (\alpha\Gamma_1)^{-1/\alpha} \leq x)e^{-x}dx \\ &= e^{-\lambda}\left(\int_0^\infty e^{-\alpha^{-1}(x+\lambda)^{-\alpha}-x}dx - e^{-\alpha^{-1}\lambda^{-\alpha}}\right). \end{aligned}$$

The result then holds since for every $x > 0$, $\exp(-\alpha^{-1}(x+\lambda)^{-\alpha} - x) \sim e^{-x}$ as $\lambda \to \infty$. We will now consider the second claim. Clearly, for each i, Γ_i is a Gamma random variable with parameters $(i, 1)$. By conditioning on the exponential random variable E_i,

$$\begin{aligned} E[\|Y_i\|^\alpha e^{\|Y_i\|}] &= \frac{1}{\Gamma(i)}\int_0^\infty \int_{\alpha^{-1}x^\alpha}^\infty \alpha^{-1}y^{i-2}e^{(\alpha y)^{-1/\alpha}-y}dye^{-x}dx \\ &= \frac{1}{\alpha\Gamma(i)}\int_0^\infty y^{i-2}e^{(\alpha y)^{-1/\alpha}-y}\int_{(\alpha y)^{-1/\alpha}}^\infty e^{-x}dxdy \\ &= \frac{1}{\alpha\Gamma(i)}\int_0^\infty y^{i-2}e^{-y}dy = \frac{\Gamma(i-1)}{\alpha\Gamma(i)}, \end{aligned}$$

which gives the result.

The Markov inequality gives $P(\|Y_i\| \geq \lambda) = o(e^{-\lambda}\lambda^{-\alpha})$ as $\lambda \to \infty$ for $i \geq 2$. Hence, the first term Y_1 dominates the rest. It is a natural to conjecture that $P\left(\|\sum_{i=1}^\infty Y_i\| \geq \lambda\right) \sim C\lambda^{-\alpha}e^{-\lambda}$ as $\lambda \to \infty$, where C is a suitable constant.

2.6 Fitting to Asset Prices

Let us justify the applicability of tempered stable distributions as a model for actual asset price processes. Assume that an asset price process $\{S_t : t \in [0,T]\}$ in $\mathbb{R}$ is given by

$$S_t = S_0 \exp(X_t),$$

where $\{X_t : t \geq 0\} \sim TS(\alpha, \rho; \gamma)$. We will consider the stock price processes of four companies, TOYOTA, SONY, HONDA and YAMAHA of the period 4/15/1998 to 4/15/2003. This period consists of 1,232 business days. We discretize the model $\{S_{t_n}\}_{n=1}^{1232}$ by $S_{t_n} = S_0 \exp(\sum_{i=1}^{n} X_{t_i})$, i.e., $\{X_{t_n}\}_{n=1}^{1231}$ are the log increments $X_{t_n} = \ln(S_{t_n}/S_{t_{n-1}})$. By the independence and stationarity of increments, $\{X_{t_n}\}_{n=1}^{1231}$ is a sequence of iid tempered stable random variables. Denote by $\Delta := 1/246$ the time increment, i.e., around 246 business days per year.

We first construct a histogram $\{n_i\}_{i=1}^{N}$ of $\{X_{t_n}\}_{n=1}^{1231}$ on N equidistant intervals over a compact domain, where n_i is the number of sample points in the ith interval. Let us call the normalized version $\widehat{n}_i$ the *binned data*. In the Black-Scholes model, such densities are assumed to be Gaussian, and its mean and variance are estimated directly from the binned data. The binned data points ($\circ$) and the fitted Gaussian density (- -) are given in Figure 2.4 together.

Let us fit the tempered stable distributions. By Theorem 2.2.7, it suffices to specify α, ρ and γ. The estimation of γ is straightforward since $\widehat{\gamma} = E[X]$ is an unbiased estimator of γ as in the Gaussian case. On the other hand, the estimation of the stability index α is not as simple. Several methods for the stability index of α-stable distributions have been proposed. (See Weron [47] for thorough comparison.) They are essentially all based upon the method of moments. But since there are only a finite number of increments, marginals necessarily have a finite second moment, and estimated α's tend to be very close to 2. We instead use an estimator introduced by

Cohen and Istas [10], based on the short time behavior of tempered stable processes in Theorem 2.3.4. Let $\#B$ represent the cardinality of the set B.

Proposition 2.6.1. *Let $\{X_t : t \geq 0\} \sim TS(\alpha, \rho; 0)$ with $\int_{\mathbb{R}_0} x^2 \rho(dx) < \infty$. For $n \in \mathbb{N}$ and $t_0 \in [0, t]$, set $A_n := \{k \in \mathbb{Z} : |k/2^n - t_0| \leq \epsilon_n\}$ where $\epsilon_n > 2^{-n}$ and $\epsilon_n \to 0$ as $n \to \infty$. Then,*

$$-(n(\#A_n))^{-1} \sum_{k \in A_n} \log_2 |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}| \to 1/\alpha \quad a.s.$$

as $n \to \infty$.

Proof. It suffices to show that

(i) for each $t_0 > 0$, there exists a random variable Z_{t_0} such that

$$\frac{X_{(1+h)t_0} - X_{t_0}}{(ht_0)^{1/\alpha}} \xrightarrow{d} Z_{t_0} \quad as\ h \to 0,$$

and that

(ii) For $0 \leq s < t < \infty$, $(\log_2 |X_t - X_s|)^2$ is uniformly integrable.

Theorem 2.3.4 directly applies to (i), with $Z_{t_0} = t_0^{-1/\alpha} X_{t_0}^{\alpha}$ where $\{X_t^{\alpha}\}$ is a corresponding α-stable process. For (ii), set $f(x) = e^2 x 1_{\{x \leq 1\}} + e^{2\sqrt{x}} 1_{\{x > 1\}}$. Then, we have

$$E[f((\log_2 |X_t - X_s|)^2)] \leq e^2 + E|X_t - X_s|^2 < \infty.$$

Since $f(x)/x \to \infty$ as $x \to \infty$, $(\log_2 |L_t^H - L_s^H|)^2$ is uniformly integrable. The proof is complete.

Let us verify the performance of the estimator. Fix the inner measure ρ in two ways; symmetric measure $\rho_1(dx) = \delta_{-1.0}(dx) + \delta_{1.0}(dx)$ and asymmetric one $\rho_2(dx) = 0.5^{-\alpha}\delta_{-0.5}(dx) + \delta_{1.0}(dx)$. For each ρ, we set $\alpha = 0.8$, 1.4, 1.7, and 1.9. We also check precision in fineness of sample paths (i.e., 2^n in Proposition 2.6.1). We generate $K = 30$ independent sample paths by the series representation of Theorem 2.4.1 and obtain the estimates $\{\alpha_k\}_{k=1}^K$. In the tables below, the mean $\widehat{\alpha} = K^{-1}\sum_{k=1}^K \alpha_k$, the standard deviation σ where

$$\sigma^2 := \frac{1}{K-1}\left(\sum_{k=1}^K \alpha_k^2 - \frac{1}{K}\left(\sum_{k=1}^K \alpha_k\right)^2\right),$$

and the mean squared error

$$\mathrm{MSE} = \frac{1}{K}\sum_{k=1}^K (\alpha_k - \alpha)^2,$$

are given.

Table 2.1: Estimation results for α with symmetric ρ_1

2^n	2^6	2^7	2^8	2^9	2^6	2^7	2^8	2^9
	$\alpha = 0.8$				$\alpha = 1.4$			
$\widehat{\alpha}$	0.9109	0.9089	0.9000	0.8917	1.4055	1.3962	1.3986	1.4006
σ	0.0426	0.0247	0.0114	0.0057	0.0754	0.0442	0.0236	0.0190
$\sqrt{\mathrm{MSE}}$	0.0140	0.0124	0.0101	0.0084	0.0054	0.0019	0.0005	0.0003
	$\alpha = 1.7$				$\alpha = 1.9$			
$\widehat{\alpha}$	1.6638	1.6643	1.6836	1.6561	1.8866	1.8478	1.8388	1.8245
σ	0.1128	0.0508	0.0345	0.0242	0.1319	0.0822	0.0473	0.0264
$\sqrt{\mathrm{MSE}}$	0.0134	0.0037	0.0014	0.0025	0.0167	0.0092	0.0059	0.0064

Table 2.2: Estimation results for α with asymmetric ρ_2

2^n	2^6	2^7	2^8	2^9	2^6	2^7	2^8	2^9
	$\alpha = 0.8$				$\alpha = 1.4$			
$\widehat{\alpha}$	0.9159	0.9022	0.8971	0.8930	1.4034	1.3962	1.4068	1.3994
σ	0.0408	0.0262	0.0158	0.0083	0.0580	0.0412	0.0223	0.0144
$\sqrt{\mathrm{MSE}}$	0.0150	0.0111	0.0097	0.0087	0.0032	0.0016	0.0005	0.0002
	$\alpha = 1.7$				$\alpha = 1.9$			
$\widehat{\alpha}$	1.6518	1.6710	1.6562	1.6448	1.8820	1.8773	1.8331	1.8143
σ	0.0803	0.0583	0.0331	0.0199	0.1371	0.0531	0.0476	0.0320
$\sqrt{\mathrm{MSE}}$	0.0085	0.0041	0.0030	0.0034	0.0182	0.0032	0.0066	0.0083

As seen, regardless of symmetry, the stability index seems to be estimated well, especially in cases of $\alpha \sim 1.4$. Some underestimation can be observed for $\alpha > 1.4$, while overestimation is apparent for $\alpha < 1.4$. The fineness 2^8 seems to give sufficiently accurate estimates. The fineness $2^8 (= 256)$ is roughly equivalent to daily stock price data (246 business days a year ($t = 1$).)

For TOYOTA, SONY, HONDA and YAMAHA, the estimated α's are 1.8573, 1.8364, 1.8157 and 1.7920, respectively. It thus suffices to consider the case of $\alpha > 1$. Then, the characteristic function of a tempered stable distribution in $\mathbb{R}$ with $\alpha > 1$ is given by

$$\widehat{\mu}(y) = \exp\left[\Gamma(-\alpha)\int_{\mathbb{R}_0}\big((1 - iyx)^{\alpha} - 1 + i\alpha yx\big)\rho(dx) + iy\gamma\right].$$

Assume that $\rho(\{x : x > 0.5\}) = 0$ and ρ is absolutely continuous with respect to the Lebesgue measure and moreover that it possess a polynomial density on the support. Then, the density of ρ can be estimated by the ordinary least square fitting. With estimated γ, α and ρ, we can generate a density of tempered stable distributions by the Fourier inversion. They are drawn in Figure 2.4. Tempered stable laws capture the outlines of the binned data very well, especially of the near-zero peakness, far better than Gaussian law. Moreover, the tails of tempered stable distributions capture outliers, as seen in Figure 2.2.

2.7 Concluding Remarks

We have studied many interesting properties of tempered stable distributions and processes. They nicely capture some key features of the statistical behaviors of financial asset prices, for example, the non-Gaussian marginals, the local spatiotemporal fractality and the global aggregational Gaussianity.

The author became aware of the new preprint [42] of Rosiński at the last stage of writing this thesis. In the preprint, the Lévy measure of tempered stable distributions is defined by the polar-coordinate form (2.2.3) rather than ours (2.2.1). The closeness under stochastic integrations, the series representation via the rejection method, the tail behaviors and all the numerical analysis are new in this thesis.

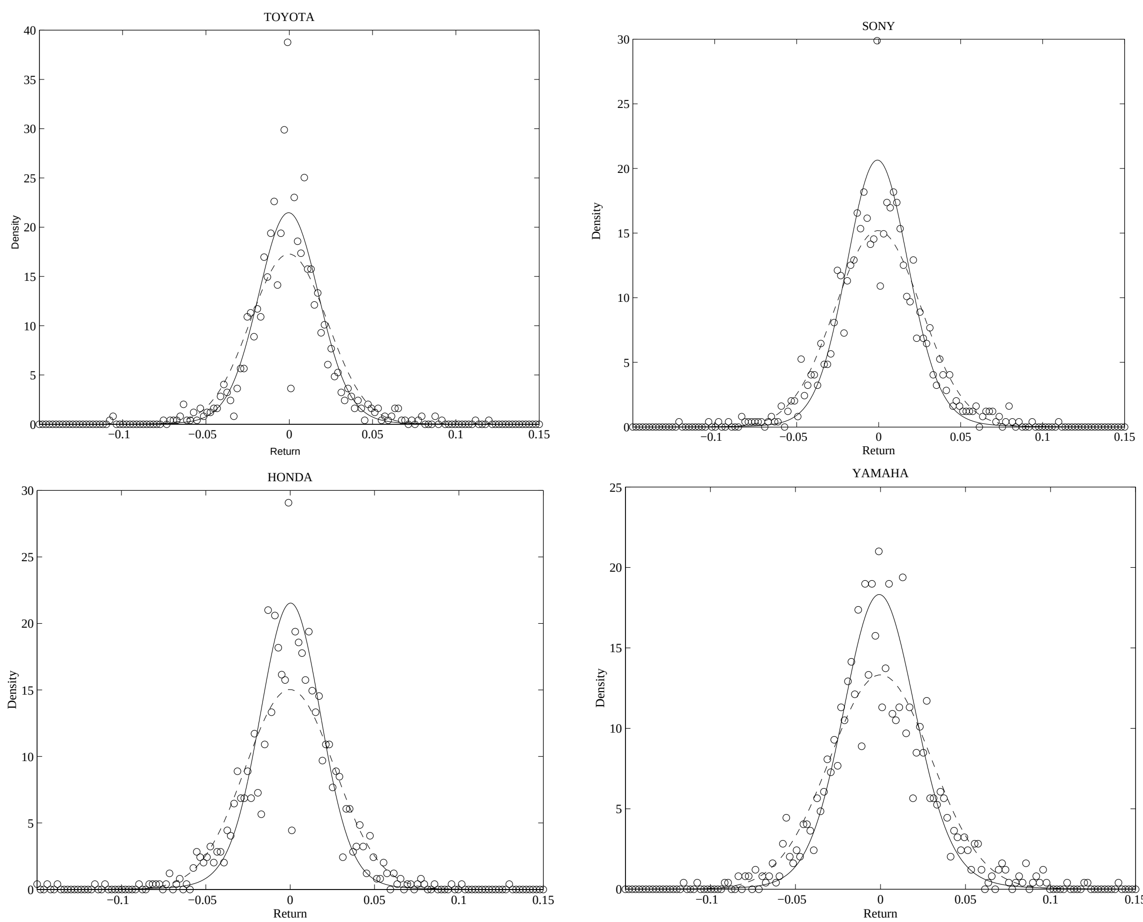


Figure 2.4: Binned data (∘), tempered stable fit (—), and Gaussian fit (- -)

Chapter 3

Fractional Tempered Stable Motions

It has often been claimed that asset prices exhibit *long-range dependence.* (See, for example, Mandelbrot [28] and Willinger *et al.* [48].) To model the property, *fractional Brownian motion (fBm)* has widely been used. The concept of fBm was first introduced as early as in 1940 by Kolmogorov [22]; Fractional Brownian motion $\{B_t^H : t \in \mathbb{R}\}$, $H \in (0, 1]$ is a centered Gaussian process with the following covariance structure;

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t, s \in \mathbb{R}. \tag{3.0.1}$$

An integral representation via a Volterra kernel (to be defined later) is introduced by Decreusefond and Üstunel [11]:

$$B_t^H = \int_0^t K_H(t, s) dW_s.$$

Some other representations of fBm can also be found in the literature. All representations are equal in law due to the Gaussianity of fBm. On the other hand, the class of *fractional Lévy motions (fLm)* (sometimes called *fractional stable motions*) has been introduced as a replacement of fBm to generate higher variation which Gaussian processes cannot. (See, for example, Samorodnitsky and Taqqu [45].)

In this chapter, we define and study a new class of *fractional tempered stable motions (fTSm)* with a view towards financial modeling. Their behaviors capture the asset price dynamics very well in that;

(i) they exhibit long-range dependence.

(ii) their marginals have heavier tails than Gaussian distribution and lighter tails than stable distributions.

(iii) they possess asymptotic selfsimilarity; they behave like a fractional stable motion in a short period of time while they are approximately fractional Brownian motions in long time.

Moreover, just like fractional Brownian motions, fractional tempered stable motions with $H \in (1/\alpha, 1)$ have a.s. Hölder continuous non-semimartingale paths. We provide series representations for simulation and discuss parameter estimations for practical use.

3.1 Definition of Fractional Tempered Stable Motions

We will define fractional tempered stable motions via tempered stable processes discussed in the preceding chapter. Recall that a probability measure μ is called tempered stable if it is infinitely divisible without Gaussian component with Lévy measure of the form

$$\nu(B) = \int_{\mathbb{R}_0^d} \int_0^\infty 1_B(sx) s^{-\alpha-1} e^{-s} ds \rho(dx), \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where $\alpha \in (0, 2)$ and the measure ρ on $\mathbb{R}^d$ is called an inner measure, satisfying

$$\int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) < \infty \quad \text{and} \quad \rho(\{0\}) = 0.$$

Moreover, assume that $\int_{\mathbb{R}_0^d} \|x\|^2 \rho(dx) < \infty$ and further that $\int_{\mathbb{R}_0^d} \|x\| \rho(dx) < \infty$ when $\alpha \in (0, 1)$. These conditions ensure that for every $\alpha \in (0, 2)$, μ has finite first and second moments. Then, for $\alpha \in (0,2)$, the characteristic function $\widehat{\mu}$ is given by

$$\widehat{\mu}(y) = \exp\left[\int_{\mathbb{R}_0^d} \phi_\alpha(\langle y, x\rangle)\rho(dx) + i\langle y, \gamma\rangle\right],$$

with some $\gamma \in \mathbb{R}^d$ and where

$$\phi_\alpha(s) = \begin{cases} \Gamma(-\alpha)((1-is)^\alpha - 1 + i\alpha s), & \text{if } \alpha \neq 1, \\ (1-is)\ln(1-is) + is, & \text{if } \alpha = 1. \end{cases} \tag{3.1.1}$$

Note that $\int_{\mathbb{R}_0^d} \|x\|^2 \rho(dx) < \infty$ implies $\int_{\mathbb{R}_0^d} \|x\|(1 + \ln^+ \|x\|)\rho(dx) < \infty$.

Throughout this chapter, we denote by $\{X_t^{TS} : t \geq 0\}$ a tempered stable Lévy process in $\mathbb{R}$ where the characteristic function of X_1^{TS} is given by the above with γ vanished, i.e.,

$$E[e^{i\langle y, X_1^{TS}\rangle}] = \exp\left[\int_{\mathbb{R}_0} \phi_\alpha(\langle y, x\rangle)\rho(dx)\right]. \tag{3.1.2}$$

Notice that by vanishing γ, we get $E[X_t^{TS}] = 0$ for every $t \geq 0$, and thus $\{X_t^{TS} : t \geq 0\}$ is a martingale. Moreover, we have $E[(X_t^{TS})^2] = t\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 \rho(dx)$. In view of Theorem 2.2.7, two parameters α and ρ in (3.1.2) uniquely characterize tempered stable distributions. Moreover, note that

$$\mathcal{L}(X_1^{TS}) \sim \begin{cases} TS(\alpha, \rho; \gamma_1), & \text{if } \alpha \in (0,1), \\ TS(\alpha, \rho; 0), & \text{if } \alpha \in [1,2), \end{cases}$$

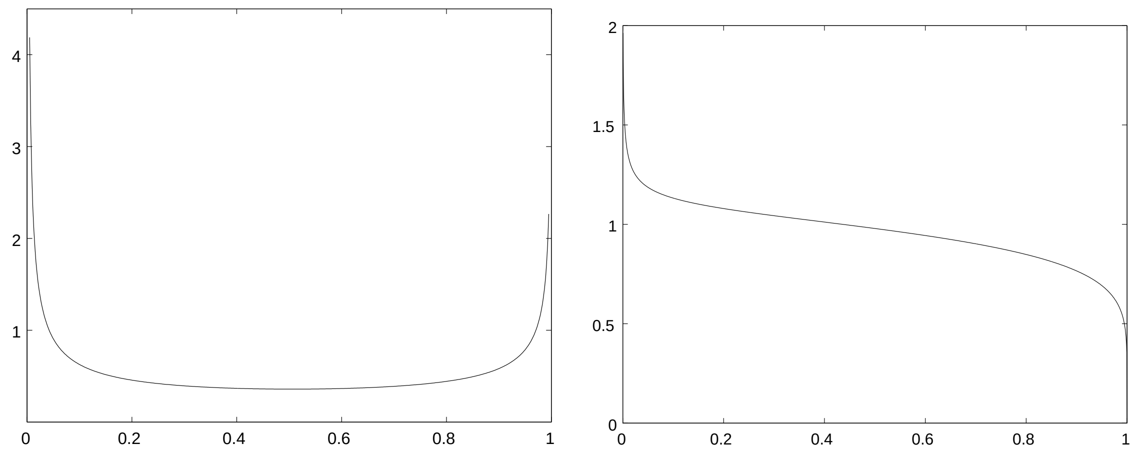
where $\gamma_1 = \Gamma(1-\alpha)\int_{\mathbb{R}_0} x\rho(dx)$. To proceed to the definition of fTSm, let us define a Volterra kernel. Recall that by a "Volterra kernel" we mean a function of two variables whose value is zero whenever the second argument is greater than the first one. Here, a Volterra kernel $K_{H,\alpha} : [0,\infty) \times [0,\infty) \to [0,\infty)$ is defined by

$$\begin{aligned} K_{H,\alpha}(t,s) \;:=\; c_{H,\alpha}\Bigg[& \left(\frac{t}{s}\right)^{H-1/\alpha} (t-s)^{H-1/\alpha} \\ & - \left(H - \frac{1}{\alpha}\right) s^{1/\alpha - H} \int_s^t u^{H-1/\alpha-1}(u-s)^{H-1/\alpha} du\Bigg] 1_{[0,t]}(s), \end{aligned}$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$, $\alpha \in (0,2)$, and

$$c_{H,\alpha} = \left(\frac{G(1-2G)\Gamma(1/2-G)}{\Gamma(2-2G)\Gamma(G+1/2)} \right)^{1/2},$$

with $G = H - 1/\alpha + 1/2$. Figure 3.1 shows typical shapes of the kernel, for the cases $H < 1/\alpha$ and $H > 1/\alpha$.



$H < 1/\alpha$ ($K_{1.2,0.6}(1,\cdot)$) $H > 1/\alpha$ ($K_{0.7,1.8}(1,\cdot)$)

Figure 3.1: Typical shape of the Volterra kernel

Definition 3.1.1. *A fractional tempered stable motion* $\{L_t^H : t \geq 0\}$ *in* $\mathbb{R}$ *is defined by*

$$L_t^H := \int_0^t K_{H,\alpha}(t,s) dX_s^{TS}, \quad t \geq 0, \tag{3.1.3}$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$.

When (3.1.3) holds, we will write $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Note that $L_\cdot^{1/\alpha} = X_\cdot^{TS}$ because $K_{1/\alpha,\alpha}(t,s) \equiv 1$ for $s \in [0,t]$. The range of the parameter H ensures that for each $t \geq 0$, $K_{H,\alpha}(t,\cdot) \in L^2([0,t])$. Below, we will also derive the necessary and sufficient condition of H so that the kernel is in $L^p([0,t])$.

Lemma 3.1.2. *If* $H \in (\max(0, 1/\alpha - 1/p), 1/\alpha + 1/p)$, *then* $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$. *In particular, if* $H \in (0, 2/\alpha)$, $K_{H,\alpha}(t,\cdot) \in L^\alpha([0,t])$, *and if* $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$,

$K_{H,\alpha}(t,\cdot) \in L^2([0,t])$. Moreover, if $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$, then

$$\int_0^t (K_{H,\alpha}(t,s))^p ds = C_p \, t^{p(H-1/\alpha)+1}, \tag{3.1.4}$$

where

$$C_p = c_{H,\alpha}^p \int_0^1 v^{p\left(\frac{1}{\alpha}-H\right)} \left[(1-v)^{H-\frac{1}{\alpha}} - \left(H - \frac{1}{\alpha}\right) \int_v^1 w^{H-\frac{1}{\alpha}-1}(w-v)^{H-\frac{1}{\alpha}} dw\right]^p dv. \tag{3.1.5}$$

Proof. When $H = 1/\alpha$, $K_{H,\alpha}(t,s) \equiv 1$ on $s \in [0,t]$. Thus, let us consider the case $H \neq 1/\alpha$. If $H > 1/\alpha$, then $K_{H,\alpha}(t,s)$ is decreasing in s and $K_{H,\alpha}(t,s) \sim C' s^{1/\alpha - H}$ as $s \downarrow 0$ for some constant C'. Hence, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if $p(1/\alpha - H) > -1$, i.e., $H < 1/\alpha + 1/p$. If $H < 1/\alpha$, then $K_{H,\alpha}(\cdot,s)$ explodes at $s = 0$ and $s = t$. We can show that $K_{H,\alpha}(t,s) \sim C'' s^{H-1/\alpha}$ as $s \downarrow 0$ and $K_{H,\alpha}(t,s) \sim C'''(t-s)^{H-1/\alpha}$ as $s \uparrow t$ for some constants C'' and C'''. Thus, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if $p(H - 1/\alpha) > -1$, i.e., $H > 1/\alpha - 1/p$, which proves the first claim. The second claim is obtained by the elementary calculus.

The characteristic function reveals that marginal law of fTSm is tempered stable.

Proposition 3.1.3. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Then, for each $t > 0$,*

$$E[e^{iyL_t^H}] = \exp\left[\int_0^t \int_{\mathbb{R}_0} \phi_\alpha(yxK_{H,\alpha}(t,s))\rho(dx)ds\right]. \tag{3.1.6}$$

In particular, for each $t \geq 0$,

$$\mathcal{L}(L_t^H) \sim TS(\alpha, \eta_t; \gamma_t) \tag{3.1.7}$$

where $\eta_t = G \circ J_t$ with $G(dx,ds) = \rho(dx)ds$ and $J_t(B) = \{(x,s) \in \mathbb{R}_0 \times [0,t] : xK_{H,\alpha}(t,s) \in B\}$ for $B \in \mathcal{B}(\mathbb{R}_0)$, and

$$\gamma_t = \begin{cases} \Gamma(1-\alpha) \int_{\mathbb{R}_0} x\rho(dx) \int_0^t K_{H,\alpha}(t,s)ds, & if \ \alpha \in (0,1) \\ 0, & if \ \alpha \in [1,2), \end{cases}$$

Proof. Since $K_{H,\alpha}(t,\cdot) \in L^\alpha([0,t])$, Proposition 2.3.2 implies that

$$E[e^{iyL_t^H}] = \exp\left[\int_{\mathbb{R}_0} \phi_\alpha(yx)\eta_t(dx)\right],$$

which gives (3.1.6). Moreover, the measure η_t is well defined as an inner measure because

$$\int_{\mathbb{R}_0} |x|^\alpha \eta_t(dx) = \int_{\mathbb{R}_0} |x|^\alpha \rho(dx) \int_0^t (K_{H,\alpha}(t,s))^\alpha ds < \infty,$$

which proves (3.1.7).

Let us close this section by stating an obvious, but very important, property of the kernel.

Lemma 3.1.4. *For each $h > 0$,*

$$K_{H,\alpha}(ht,s) = h^{H-1/\alpha} K_{H,\alpha}(t,s/h).$$

3.2 Covariance Structure and Long-Range Dependence

For notational convenience, we will henceforth set $G := H - 1/\alpha + 1/2$.

Proposition 3.2.1. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H,\alpha,\rho)$. Then, $E[L_t^H] = 0$, and*

$$\mathrm{Cov}(L_t^H, L_s^H) = \frac{1}{2}\left(t^{2G} + s^{2G} - (t-s)^{2G}\right) E[(X_1^{TS})^2], \quad s \in [0,t]. \tag{3.2.1}$$

Proof. By Lemma 3.1.2, if $H \in (\max(0, 1/\alpha - 1), 1/\alpha + 1)$, then $E[|L_t^H|] < \infty$. Since $\{X_t^{TS} : t \geq 0\}$ is a centered martingale, we get $E[L_t^H] = 0$. For the second claim, recall first that $\{X_t^{TS} : t \geq 0\}$ has finite second moment for each $t \geq 0$. Since $K_{H,\alpha}(t,\cdot) \in L^2([0,t])$, we have $E[|L_t^H|^2] < \infty$ and thus covariance is well defined. Since $\{L_t^H : t \geq 0\}$ is centered, we have for $s \in [0,t]$,

$$\begin{aligned}
\mathrm{Cov}(L_t^H, L_s^H) &= E[L_t^H L_s^H] = E\left[\int_0^t K_{H,\alpha}(t,u)dX_u^{TS} \int_0^s K_{H,\alpha}(s,u)dX_u^{TS}\right] \\
&= E\left[\int_0^s K_{H,\alpha}(t,u)dX_u^{TS} \int_0^s K_{H,\alpha}(s,u)dX_u^{TS}\right] \\
&= E[(X_1^{TS})^2] \int_0^s K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)du,
\end{aligned}$$

where the last equality holds by the Itô isometry. Thus, it is enough to show that

$$\int_0^s K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)du = \frac{1}{2}(t^{2G} + s^{2G} - (t-s)^{2G}). \tag{3.2.2}$$

When $H > 1/\alpha$, the kernel can be written as

$$K_{H,\alpha}(t,s) = c_{H,\alpha}(G-1/2)s^{1/2-G}\int_s^t (v-s)^{G-3/2}v^{G-1/2}dv.$$

Then, we have

$$\begin{aligned}
&((G-1/2)c_{H,\alpha})^{-2}\int_0^s K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)du \\
&\quad = \int_0^s u^{1-2G}\int_u^t\int_u^s (vw)^{G-1/2}(v-u)^{G-3/2}(w-u)^{G-3/2}dw\,dv\,du \\
&\quad = \int_0^t\int_0^s (vw)^{G-1/2}\int_0^w u^{1-2G}(v-u)^{G-3/2}(w-u)^{G-3/2}du\,dw\,dv \\
&\quad = \int_0^t\int_0^s (vw)^{G-1/2}|v-w|^{2G-2}\int_{v/w}^\infty (xw-v)^{1-2G}x^{G-3/2}dx\,dw\,dv \\
&\quad = \int_0^1 (1-y)^{1-2G}y^{G-3/2}dy\int_0^t\int_0^s |v-w|^{2G-2}dw\,dv \\
&\quad = \frac{B(2-2G,G-1/2)}{G(2G-1)}\frac{1}{2}(t^{2G}+s^{2G}-(t-s)^{2G}),
\end{aligned}$$

where the second equality holds by Fubini theorem, the third by the change of variables $x = (v-u)/(w-u)$ and the fourth by $y = t/(sx)$. We will consider the case $H < 1/\alpha$. As in Nualart [34], we prove the equation of (3.2.2) differentiated with respect to t. Observing that

$$\partial_1 K_{H,\alpha}(t,u) = c_{H,\alpha}(G-1/2)\left(\frac{t}{u}\right)^{G-1/2}(t-u)^{G-3/2},$$

we have

$$
\begin{aligned}
&\int_0^s \partial_1 K_{H,\alpha}(t,u) K_{H,\alpha}(s,u)du \\
&\quad = c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)\int_0^s \left(\frac{t}{u}\right)^{G-\frac{1}{2}} (t-u)^{G-\frac{3}{2}} \left(\frac{s}{u}\right)^{G-\frac{1}{2}} (s-u)^{G-\frac{1}{2}} dv\, du \\
&\qquad\quad - c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)^2 \int_0^s \left(\frac{t}{u}\right)^{G-\frac{1}{2}} (t-u)^{G-\frac{3}{2}} u^{\frac{1}{2}-G} \int_u^s v^{G-\frac{3}{2}} (v-u)^{G-\frac{1}{2}} dv\, du \\
&\quad = c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)(ts)^{G-\frac{1}{2}} \int_0^s u^{1-2G}(t-u)^{G-\frac{3}{2}}(s-u)^{G-\frac{1}{2}} du \\
&\qquad\quad - c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)^2 t^{G-\frac{1}{2}} \int_0^s v^{G-3/2} \int_0^v u^{1-2G}(t-u)^{G-\frac{3}{2}}(v-u)^{G-\frac{1}{2}} du\, du,
\end{aligned}
$$

by the Fubini theorem. Then, by the change of variables $x=(t-u)/(s-u)$ for the first integral of the last term and $y=(t-u)/(v-u)$ for the second, we get

$$
\begin{aligned}
&c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)(ts)^{G-\frac{1}{2}} \int_0^1 x^{1-2G}\left(\frac{t}{s}-x\right)^{G-\frac{3}{2}} (1-x)^{G-\frac{1}{2}} dx \\
&\qquad\quad - c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)^2 t^{G-\frac{1}{2}} \int_0^s v^{G-\frac{3}{2}} \int_0^1 y^{1-2G}\left(\frac{t}{v}-y\right)^{G-\frac{3}{2}} (1-y)^{G-\frac{1}{2}} dy\, dv \\
&\quad = c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)(ts)^{G-\frac{1}{2}} \int_0^1 x^{1-2G}\left(\frac{t}{s}-x\right)^{G-\frac{3}{2}} (1-x)^{G-\frac{1}{2}} dx \\
&\qquad\quad - c_{H,\alpha}^2 \left(G-\frac{1}{2}\right) t^{G-\frac{1}{2}} \int_0^1 y^{-2G}(1-y)^{G-\frac{1}{2}} dy \\
&\qquad\qquad + c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)(ts)^{G-\frac{1}{2}} \int_0^1 y^{-2G}\left(\frac{t}{s}-y\right)^{G-\frac{1}{2}} (1-y)^{G-\frac{1}{2}} dy \\
&\quad = c_{H,\alpha}^2 \left(G-\frac{1}{2}\right)(ts)^{G-\frac{1}{2}} \frac{t}{s} \int_0^1 x^{-2G}\left(\frac{t}{s}-x\right)^{G-\frac{3}{2}} (1-x)^{G-\frac{1}{2}} dx \\
&\qquad\quad - c_{H,\alpha}^2 \left(G-\frac{1}{2}\right) t^{G-\frac{1}{2}} \int_0^1 y^{-2G}(1-y)^{G-\frac{1}{2}} dy,
\end{aligned}
$$

which equals $G(t^{2G}-(t-s)^{2G})$. The proof is complete.

Corollary 3.2.2. *For each* $t \geq 0$,

$$\mathrm{Var}(L_t^H) = t^{2G} E[(X_1^{TS})^2].$$

Moreover, for $s \in [0,t]$,

$$E[(L_t^H - L_s^H)^2] = E[(L_{t-s}^H)^2], \tag{3.2.3}$$

and for each $h > 0$,

$$E[(L_{ht}^H)^2] = E[(h^G L_t^H)^2]. \tag{3.2.4}$$

Proof. We get the first and the second, respectively, by $\mathrm{Var}(L_t^H) = \mathrm{Cov}(L_t^H, L_t^H)$ and

$$E[(L_t^H - L_s^H)^2] = \mathrm{Var}(L_t^H) + \mathrm{Var}(L_s^H) - 2\,\mathrm{Cov}(L_t^H, L_s^H) = (t-s)^{2G} E[(X_1^{TS})^2].$$

Finally, (3.2.4) is a direct consequence of the variance.

The property (3.2.4) is sometimes called *second-order* selfsimilarity. Moreover, (3.2.3) says that fTSm $\{L_t^H : t \geq 0\}$ has *second-order* stationary increments. The second-order stationary increments property derives stochastic continuity.

Corollary 3.2.3. $\{L_t^H : t \geq 0\}$ *is stochastically continuous.*

Proof. For each $\epsilon > 0$, we have

$$\begin{aligned} P(|L_t^H - L_s^H| \geq \epsilon) &\leq \epsilon^{-2} E[|L_t^H - L_s^H|^2] \\ &= \epsilon^{-2} E[|L_{t-s}^H|^2] = \epsilon^{-2}(t-s)^{2G} E[(X_1^{TS})^2], \end{aligned}$$

by (3.2.3) and the Chebyshov's inequality. Since $G > 0$, we get $\lim_{s \to t} P(|L_t^H - L_s^H| \geq \epsilon) = 0$, which concludes the proof.

We are now in a position to discuss the *long-range dependence* of fTSm. The definition of long-range dependence is often ambiguous and varies among authors. In this thesis, we say that the increments of a stochastic process $\{X_t : t \geq 0\}$ exhibit long-range dependence if for each $h > 0$,

$$\sum_{n=1}^{\infty} |\mathrm{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| = \infty,$$

or *short-range dependence*, if each $h > 0$,

$$\sum_{n=1}^{\infty} |\mathrm{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| < \infty.$$

Proposition 3.2.4. *The increments of fTSm $\{L_t^H : t \geq 0\}$ exhibit long-range dependence if $H \in (1/\alpha, 1/\alpha + 1/2)$, and short-range dependence if $H \in (1/\alpha - 1/2, 1/\alpha]$.*

Proof. By Lemma 3.2.1, we have for each $h > 0$,

$$\begin{aligned} \mathrm{Cov}(L_h^H, L_{t+h}^H - L_t^H) &= \frac{1}{2} t^{2G}((1 + h/t)^{2G} - 2 + (1 - h/t)^{2G}) \\ &\sim \frac{1}{2} t^{2(G-1)} G(2G-1)h^2 \quad as\ t \to \infty. \end{aligned}$$

Then, the claim holds because $2(G-1) > -1$ when $H \in (1/\alpha, 1/\alpha + 1/2)$, and $2(G-1) \leq -1$ when $H \in (1/\alpha - 1/2, 1/\alpha]$. The proof is complete.

In relation to the second moment, we will consider higher moments of fTSm.

Proposition 3.2.5. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Then, for each $p > 2$ and each $t \geq 0$, $E[|L_t^H|^p] < \infty$ if and only if $\int_{\mathbb{R}_0} |x|^p \rho(dx) < \infty$ and $K_{H,\alpha}(t, \cdot) \in L^p([0, t])$.*

Proof. By Proposition 3.1.3, $\mathcal{L}(L_t^H)$ is tempered stable. In view of Proposition 2.2.8, $E[|L_t^H|^p] < \infty$ if and only if $\int_{|x|>1} |x|^p \eta_t(dx) < \infty$, which is equivalent to

$$\iint_{|x|K_{H,\alpha}(t,s)>1} (|x| K_{H,\alpha}(t, s))^p \rho(dx) ds < \infty.$$

Clearly, this is further equivalent to $\int_{\mathbb{R}_0} |x|^p \rho(dx) < \infty$ and $K_{H,\alpha}(t, \cdot) \in L^p([0, t])$.

3.3 Short-time and long-time Behaviors

Definition 3.3.1. *We define an α-stable process $\{X_t^\alpha : t \geq 0\}$ in $\mathbb{R}$ by the characteristic function*

$$E[e^{iyX_1^\alpha}] = \exp\left[\int_{\mathbb{R}_0} \varphi_\alpha(yx)\rho(dx)\right].$$

where

$$\varphi_\alpha(s) = \begin{cases} -\Gamma(-\alpha) \cos \frac{\pi\alpha}{2} |s|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \mathrm{sgn}(s)), & if\ \alpha \neq 1, \\ -\frac{\pi}{2}|s|, & if\ \alpha = 1. \end{cases} \tag{3.3.1}$$

We will call $\{X_t^\alpha : t \geq 0\}$ an α-stable process *corresponding to* the tempered stable process $\{X_t^{TS} : t \geq 0\}$ under the condition that ρ is symmetric when $\alpha = 1$. By Theorem 2.3.4, we have that

$$\left\{h^{-1/\alpha}\left(X_{ht}^{TS} + b\,ht\right) : t \geq 0\right\} \xrightarrow{d} \{X_t^\alpha : t \geq 0\}, \quad \text{as } h \to 0,$$

where

$$b = \begin{cases} \Gamma(1-\alpha)\int_{\mathbb{R}_0} x\rho(dx), & \text{if } \alpha \in (0,1), \\ 0, & \text{if } \alpha \in [1,2). \end{cases} \tag{3.3.2}$$

We note that when $\alpha = 1$, centering terms vanish due to the symmetricity of the inner measure ρ. Let us now define *fractional stable motions (fSm).*

Definition 3.3.2. *Let $\{X_t^\alpha : t \geq 0\}$ be the corresponding α-stable process of $\{X_t^{TS} : t \geq 0\}$. Then, a fractional stable motion (fSm) $\{L_t^{H,\alpha}, t \geq 0\}$ corresponding to fTSm $\{L_t^H : t \geq 0\}$ is defined by*

$$L_t^{H,\alpha} := \int_0^t K_{H,\alpha}(t,s)dX_s^\alpha, \quad t \geq 0, \tag{3.3.3}$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$.

The range of the parameter H ensures $K_{H,\alpha}(t,\cdot) \in L^2([0,t])$ for each $t > 0$, and thus the integral is certainly well defined. Let us first derive basic properties of fSm.

Lemma 3.3.3. *Let $\{L_t^{H,\alpha} : t \geq 0\}$ be fSm defined as above and C_α be a constant defined as in Lemma 3.1.2.*

(i) If $\alpha \in (0,1)$, then $\{L_t^{H,\alpha} : t \geq 0\}$ is broad-sense selfsimilar;

$$\left\{h^{-H}L_{ht}^{H,\alpha} : t \geq 0\right\} \stackrel{d}{=} \left\{L_t^{H,\alpha} - (1-h^{1-1/\alpha})b\,C_\alpha t^{\alpha H} : t \geq 0\right\}.$$

If $\alpha \in [1,2)$, then $\{L_t^{H,\alpha} : t \geq 0\}$ is selfsimilar;

$$\left\{h^{-H}L_{ht}^{H,\alpha} : t \geq 0\right\} \stackrel{d}{=} \left\{L_t^{H,\alpha} : t \geq 0\right\}.$$

(ii) For each $t > 0$, the law $\mu_t \sim \mathcal{L}(L_t^{H,\alpha})$ is α-stable with characteristic function

$$\widehat{\mu}_t(y) = \exp\left[C_\alpha t^{\alpha H} \int_{\mathbb{R}_0} \varphi_\alpha(yx)\rho(dx)\right]. \tag{3.3.4}$$

Proof. (i) By Definition 3.3.1, we have

$$\{h^{-1/\alpha} X_{ht}^\alpha : t \geq 0\} \stackrel{d}{=} \left\{X_t^\alpha - (1 - h^{1-1/\alpha}) b\, t : t \geq 0\right\},$$

where b is given by (3.3.2). In view of Lemma 3.1.4, the selfsimilarity and the broad-sense selfsimilarity are preserved.

(ii) Since $K_{H,\alpha}(t, \cdot) > 0$, we have

$$\begin{aligned}
&E[e^{iyL_t^{H,\alpha}}] \\
&= \exp\Bigg[- \Gamma(-\alpha) \cos\frac{\pi\alpha}{2} \int_0^t \int_{\mathbb{R}_0} |K_{H,\alpha}(t,s)yx|^\alpha \\
&\qquad\qquad \left(1 - i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(K_{H,\alpha}(t,s)yx)\right)\rho(dx)ds\Bigg] \\
&= \exp\Bigg[- \Gamma(-\alpha) \cos\frac{\pi\alpha}{2} \int_0^t (K_{H,\alpha}(t,s))^\alpha ds \int_{\mathbb{R}_0} |yx|^\alpha \\
&\qquad\qquad \left(1 - i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(yx)\right)\rho(dx)\Bigg],
\end{aligned}$$

which equals (3.3.4) by Lemma 3.1.2. Similarly, for $\alpha = 1$,

$$\begin{aligned}
E[e^{iyL_t^{H,1}}] &= \exp\left[-\int_0^t \int_{\mathbb{R}_0} \left(\frac{\pi}{2}|K_{H,1}(t,s)yx|\right)\rho(dx)ds\right] \\
&= \exp\left[-\int_0^t K_{H,1}(t,s)ds \frac{\pi}{2}\int_{\mathbb{R}_0} |yx|\rho(dx)\right],
\end{aligned}$$

which gives (3.3.4).

We now present a main result of this section.

Theorem 3.3.4. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$, and $\{L_t^{H,\alpha} : t \geq 0\}$ the corresponding fSm. When $\alpha = 1$, assume further that ρ is symmetric. Then,*

$$\{h^{-H} L_{ht}^H - h^{1-1/\alpha} k_t : t \geq 0\} \xrightarrow{d} \{L_t^{H,\alpha} : t \geq 0\} \quad \text{as } h \to 0,$$

where $k_t = b\int_0^t K_{H,\alpha}(t,s)ds$ with b given by (3.3.2).

Proof. Observe that for each $t \geq 0$,

$$h^{-H} L_{ht}^{H} + h^{1-1/\alpha} k_t = \int_0^t K_{H,\alpha}(t,s) h^{-1/\alpha} d(X_{hs}^{TS} + b\, hs) =: Y_t^h,$$

where b is given by (3.3.2). It is then sufficient to show that for any real sequence $\{a_i\}_{i=1}^k$ and nonnegative nondecreasing real sequence $\{t_i\}_{i=1}^k$ where $k \in \mathbb{N}$, the random variable $\sum_{i=1}^k a_i Y_{t_i}^h$ converges in law to $\sum_{i=1}^k a_i L_{t_i}^{H,\alpha}$ as $h \to 0$. Since

$$\sum_{i=1}^k a_i Y_{t_i}^h = \int_0^{t_k} \left(\sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) h^{-1/\alpha} d(X_{hs}^{TS} + b\, hs),$$

we have by Proposition 3.1.3 that

$$E\left[e^{iy \sum_{i=1}^k a_i Y_{t_i}^h} \right] = \exp\left[\int_0^{t_k} \int_{\mathbb{R}_0} h \psi_\alpha (yxh^{-1/\alpha} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)) \rho(dx) ds \right],$$

where ψ_α is given by (2.2.10). In view of the proof of Theorem 2.3.4 with

$$\left| \int_0^{t_k} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) ds \right|^\alpha \leq \sum_{i=1}^k |a_i|^\alpha \int_0^{t_i} K_{H,\alpha}(t_i, s)^\alpha ds < \infty,$$

we can interchange the limit and the integral and thus we get

$$\lim_{h \to 0} E\left[e^{iy \sum_{i=1}^k a_i Y_{t_i}^h} \right] = \exp\left[\int_0^{t_k} \int_{\mathbb{R}_0} \varphi_\alpha (yx \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)) \rho(dx) ds \right],$$

where φ_α is given by (3.3.1). The right hand side is indeed the characteristic function of $\sum_{i=1}^k a_i L_{t_i}^{H,\alpha}$. The proof is complete.

Theorem 3.3.4 reads that in a short period of time, fTSm asymptotically is self-similar and, moreover, behaves like fSm. In the next theorem, we present another important result of this section. Analogously to the long-time behavior of tempered stable processes (Theorem 2.3.5), fTSm is approximately fBm in the long run.

Theorem 3.3.5. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Then,*

$$\{h^{-G} L_{ht}^H : t \geq 0\} \xrightarrow{d} \{c_B B_t^G : t \geq 0\} \quad \text{as } h \to \infty, \tag{3.3.5}$$

where $\{B_t^G : t \geq 0\}$ is a (standard) fBm with Hurst parameter G and $c_B^2 = \Gamma(2 - \alpha) \int_{\mathbb{R}_0} x^2 \rho(dx)$.

Proof. First note that for each $h > 0$,

$$\mathrm{Cov}(h^{-G}L^H_{ht}, h^{-G}L^H_{hs}) = \frac{1}{2}\left[t^{2G} + s^{2G} - (t-s)^{2G}\right] E[(X_1^{TS})^2], \quad s \in [0,t].$$

Therefore, we only need to show that the marginal law at any time of $\{h^{-G}L^H_{ht} : t \geq 0\}$ converges to Gaussian. Without loss of generality, we can fix $t = 1$. Then, by Lemma 3.1.4,

$$\begin{aligned} E[e^{iyh^{-G}L^H_h}] &= \exp\left[\int_0^h \int_{\mathbb{R}_0} \vartheta_\alpha(h^{-G}yxK_{H,\alpha}(h,s))\rho(dx)ds\right] \\ &= \exp\left[\int_0^1 \int_{\mathbb{R}_0} \vartheta_\alpha(h^{-1/2}yxK_{H,\alpha}(1,s))\rho(dx)ds\right], \end{aligned}$$

where $\vartheta_\alpha(u) = \int_0^\infty (e^{ius} - 1 - ius)s^{-\alpha-1}e^{-s}ds$. Moreover, we have, as in the proof of Theorem 2.3.5,

$$|\vartheta_\alpha(yxK_{H,\alpha}(1,s))| \leq (yx)^2\Gamma(2-\alpha)\int_0^1 (K_{H,\alpha}(1,s))^2 ds,$$

which justifies the passage to the limit and

$$\lim_{h\to\infty} E[e^{iyh^{-G}L^H_h}] = \exp\left[-\frac{y^2}{2}\int_0^1 (K_{H,\alpha}(1,s))^2 ds\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2\rho(dx)\right].$$

The proof is complete.

Willinger *et al.* [48] states that a numerical analysis of stock price time series indicates long-range dependence and their marginal distributions have heavier tails than Gaussian ones, but still with finite variance. Besides those, it has widely been known that they have heavier tails in a short period of time and are almost Gaussian in the long run. Indeed, we have seen that fTSm possess all those properties. It is thus natural to expect fTSm to better model the asset price dynamics than tempered stable processes or fBm.

We have seen in Lemma 3.3.3 that the marginals of fSm $\{L_t^{H,\alpha} : t \geq 0\}$ are α-stable. This implies that we cannot define the covariance of fSm. Instead, in the literature one can find the *covariation* for two jointly symmetric α-stable random variables X and Y with $\alpha > 1$;

$$\tau(X, Y) := \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X - Y\|_\alpha^\alpha, \tag{3.3.6}$$

where the norm $\|\cdot\|$ gives the scale of parameter, i.e. for $Z \sim S_\alpha(\sigma, 0, 0)$, $\|Z\|_\alpha = \sigma$, or more generally, the *codifference* for any jointly infinitely divisible random variables X and Y;

$$I(\theta_1, \theta_2; X, Y) := -\ln E[e^{i(\theta_1 X + \theta_2 Y)}] + \ln E[e^{i\theta_1 X}] + \ln E[e^{i\theta_2 Y}], \tag{3.3.7}$$

for $\theta_1, \theta_2 \in \mathbb{R}$. Clearly, (3.3.6) is a special case of (3.3.7). For fSm $\{L_t^{H,\alpha} : t \geq 0\}$ with a *symmetric* Lévy measure, it is known that

$$I(1, -1; L_t^{H,\alpha}, L_s^{H,\alpha}) = C(t^{\alpha H} + s^{\alpha H} - (t-s)^{\alpha H}), \quad 0 \leq s \leq t, \tag{3.3.8}$$

where C is some constant. It is also known that the codifference coincides covariance in the Gaussian case. For example, for fBm $\{B_t^H : t \geq 0\}$,

$$\tau(B_t^H, B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - (t-s)^{2H}) = \mathrm{Cov}(B_t^H, B_s^H), \quad 0 \leq s \leq t,$$

and

$$\tau(B_{t+1}^H - B_t^H, B_1^H - B_0^H) = \mathrm{Cov}(B_{t+1}^H - B_t^H, B_1^H - B_0^H) \sim Ct^{2H-2},$$

as $t \to \infty$. Let us look at the codifference of increments of fTSm.

Proposition 3.3.6. *Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$. Then,*

$$I(\theta_1, \theta_2; L_{t+1}^H - L_t^H, L_1^H - L_0^H) \sim C(\theta_1, \theta_2) t^{2(H-1/\alpha)-1},$$

as $t \to \infty$, where

$$C(\theta_1, \theta_2) = \frac{-ic_H \theta_1 \pi}{\Gamma(\alpha)\sin(\pi\alpha)} \int_{[0,1]\times\mathbb{R}_0} \left((1 - ix\theta_2 K_H(1, s))^{\alpha-1} - 1\right) x s^{1/\alpha - H} ds \rho(dx).$$

Proof. We have

$$I(\theta_1,\theta_2;L^H_{t+1}-L^H_t,L^H_1-L^H_0)$$
$$=\Gamma(-\alpha)\int_{\mathbb{R}\times\mathbb{R}_0}\Big(-(1-ix(\theta_1(K_{H,\alpha}(t+1,s)-K_{H,\alpha}(t,s))+\theta_2K_{H,\alpha}(1,s)))^\alpha$$
$$+(1-ix\theta_1(K_{H,\alpha}(t+1,s)-K_{H,\alpha}(t,s)))^\alpha$$
$$+(1-ix\theta_2K_{H,\alpha}(1,s))^\alpha-1\Big)ds\rho(dx).$$

By the elementary calculus, we can show that

$$K_{H,\alpha}(t+1,s)-K_{H,\alpha}(t,s)\sim c_Hs^{1/\alpha-H}t^{2(H-1/\alpha)-1},$$

as $t\to\infty$. Hence, for each $s\geq 0$,

$$-(1-ix(\theta_1(K_{H,\alpha}(t+1,s)-K_{H,\alpha}(t,s))+\theta_2K_{H,\alpha}(1,s)))^\alpha$$
$$+(1-ix\theta_1(K_{H,\alpha}(t+1,s)-K_{H,\alpha}(t,s)))^\alpha+(1-ix\theta_2K_{H,\alpha}(1,s))^\alpha-1$$
$$\sim i\alpha x\theta_1c_Hs^{1/\alpha-H}\left((1-ix\theta_2K_{H,\alpha}(1,s))^{\alpha-1}-1\right)t^{2(H-1/\alpha)-1},$$

as $t\to\infty$. Finally, the result holds by $\Gamma(-\alpha)=\frac{-\pi}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}$.

3.4 Sample Path Properties

In this section, we study sample path properties of fTSm. Let us begin with the case $H\in(1/\alpha-1/2,1/\alpha)$. To prove the unboundedness of its sample paths, we will use a series representation of fTSm, which may be of independent interest, derived from that of tempered stable processes. Proposition 2.4.1 tells us that $\{X^{TS}_t:t\in[0,T]\}$ admits the series representation;

$$\{X^{TS}_t:t\in[0,T]\}\stackrel{d}{=}\Bigg\{\sum_{i=1}^\infty\Bigg[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\wedge E_iU_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}1(T_i\leq t)$$
$$-m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha}k'\frac{t}{T}\Bigg]+B_T\,t:t\in[0,T]\Bigg\}.$$

where the equality holds for finite dimensional distributions and the convergence on the right hand side holds a.s uniformly in $t\in[0,T]$. Here, $\{\Gamma_i\}_{i\geq1}$, $\{E_i\}_{i\geq1}$, $\{U_i\}_{i\geq1}$,

$\{V_i\}_{i\geq 1}$, $\{T_i\}_{i\geq 1}$, $m(\rho)$ are defined as in Proposition 2.4.1. The constants k' and B_T are given by $k' = m(\rho)^{-\alpha}\int_{\mathbb{R}_0} x|x|^{\alpha-1}\rho(dx)$ and

$$B_T = \begin{cases} k'm(\rho)T^{-1}(\alpha/T)^{-1/\alpha}\zeta(-1/\alpha) + |\Gamma(1-\alpha)|\int_{\mathbb{R}_0} x\rho(dx), & \text{if } \alpha \neq 1, \\ T\left((\ln(m(\rho)T) + 2\gamma)\int_{\mathbb{R}_0} x\rho(dx) - \int_{\mathbb{R}_0^d} x\ln|x|\rho(dx)\right), & \text{if } \alpha = 1. \end{cases}$$

This series representation can easily be extended to fractional tempered stable motions as follows.

Proposition 3.4.1. *Let* $\{L_t^H : t \in [0,T]\} \sim fTSm(H,\alpha,\rho)$. *Then,*

$$\begin{aligned} &\{L_t^H : t \in [0,T]\} \\ &\quad \stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t,T_i) \right.\right. \\ &\quad \left.\left. - m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha} k' C_1 \frac{t^{H-1/\alpha+1}}{T} \right] + C_1 B_T t^{H-1/\alpha+1} : t \in [0,T] \right\}, \end{aligned} \tag{3.4.1}$$

where C_1 *is the constant defined by (3.1.5). Moreover, if* $H \in [1/\alpha, 1/\alpha + 1/2)$, *then the equality holds a.s. and the series converges a.s. uniformly on* $[0,T]$.

Proof. Let $\{Z_t : t \in [0,T]\}$ be a stochastic process on the same probability space as the stochastic process of the right hand side of the above, defined by

$$Z_t := \sum_{i=1}^{\infty} \left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t,T_i) - c_i(T)E[K_{H,\alpha}(t,T_1)] \right], \tag{3.4.2}$$

where

$$c_i(T) := \int_{i-1}^{i} E\left[\left(m(\rho)\left(\frac{\alpha r}{T}\right)^{-1/\alpha} \wedge E_1 U_1^{1/\alpha}|V_1| \right) \frac{V_1}{|V_1|} \right] dr.$$

By the same argument in Proposition 2.4.1, we can show that for each $t \in [0,T]$,

$$\sum_{i=1}^{\infty} \left[m(\rho)\left(\frac{\alpha\, i}{T}\right)^{-1/\alpha} k' C_1 \frac{t^{H-1/\alpha+1}}{T} - c_i(T)E[K_{H,\alpha}(t,T_1)] \right] = C_1 B_T t^{H-1/\alpha+1}.$$

We will first prove the equality of finite dimensional distributions by showing that the random variable $\sum_{j=1}^{k} a_j Z_{t_j}$ has the same law as $\sum_{j=1}^{k} a_j L_{t_j}^H$ for any real sequence

$\{a_j\}_{j=1}^k$ and nondecreasing sequence $\{t_j\}_{j=1}^k$ taking values in $[0,T]$ where $k \in \mathbb{N}$. In view of Proposition 3.1.3, we have

$$E\left[e^{iy\sum_{j=1}^k a_j L_{t_j}^H}\right] = \exp\left[\int_0^T \int_{\mathbb{R}_0} \psi_\alpha(yx\sum_{j=1}^k a_j K_{H,\alpha}(t_j,s))\rho(dx)ds\right],$$

where ψ_α is given by (2.2.10). Now, observe that

$$\sum_{j=1}^k a_j Z_{t_j} = \sum_{i=1}^\infty \left[\left(m(\rho)\left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha}|V_i|\right)\frac{V_i}{|V_i|}\sum_{j=1}^k a_j K_{H,\alpha}(t_j,T_i)\right.$$
$$\left. -c_i(T)E\left[\sum_{j=1}^k a_j K_{H,\alpha}(t_j,T_1)\right]\right].$$

This series representation is induced by the Lévy measure

$$\begin{aligned}
&\nu(B)\\
&= \int_0^T \int_{\mathbb{R}_0} \int_0^\infty \int_0^1 \int_0^\infty 1_B(H(r/T,u,s,x)\sum_{j=1}^k a_j K_{H,\alpha}(t_j,v))drdue^{-s}ds\rho_1(dx)\frac{dv}{T}\\
&= \int_0^T \int_{\mathbb{R}_0} \int_0^\infty 1_B(sx\sum_{j=1}^k a_j K_{H,\alpha}(t_j,v))s^{-\alpha-1}e^{-s}ds\rho(dx)dv,
\end{aligned}$$

where $H(r,u,s,x) = (m(\rho)(\alpha r)^{-1/\alpha} \wedge su^{1/\alpha}|x|)\cdot x/|x|)$ and $\rho_1(dx) = m(\rho)^{-\alpha}|x|^\alpha\rho(dx)$. In fact, the measure ν is well defined as a Lévy measure because $K_{H,\alpha}(t_j,\cdot) \in L^2([0,t_j])$ for each j. Therefore, by Theorem 1.4.1 (iii), we get

$$\begin{aligned}
E[e^{iy\sum_{j=1}^k a_j Z_{t_j}}] &= \exp\left[\int_{\mathbb{R}_0}(e^{iyz}-1-iyz)\nu(dz)\right]\\
&= \exp\left[\int_0^T \int_{\mathbb{R}_0} \psi_\alpha(yx\sum_{j=1}^k a_j K_{H,\alpha}(t_j,s))\rho(dx)ds\right],
\end{aligned}$$

which proves the equality of finite dimensional distributions. Finally, we prove the almost sure equality. Fix $H \in [1/\alpha, 1/\alpha+1/2)$. The almost sure equality for each $t \in [0,T]$ is straightforward by the same argument as Theorem 1.4.1 (iv). It remains to show that series converges uniformly on $[0,T]$. We will again use $\{Z_t : t \in [0,T]\}$.

Define, for $t \in [0, T]$ and $s \in [0, \infty)$,

$$Z_{t,s} := \sum_{\{i:\Gamma_i \le s\}} \left[\left(m(\rho) \left(\frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} |V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) \right.$$
$$\left. - c_i(T) E[K_{H,\alpha}(t, T_1)] \right].$$

Then, $\{Z_{t,s} : t \in [0, T], s \in [0, \infty)\}$ has independent increments in s (*not* in t, of course.) Therefore, the same argument as the proof of Theorem 5.1 of Rosiński [40] gives the a.s. convergence of the series uniformly on $[0, T]$. The proof is complete.

The reason why we did not consider the a.s. equality for $H \in (1/\alpha - 1/2, 1/\alpha)$ is the unboundedness of sample path shown below.

Theorem 3.4.2. *If $H \in (1/\alpha - 1/2, 1/\alpha)$, $\{L_t^H : t \ge 0\} \sim fTSm(H, \alpha, \rho)$ is nowhere bounded.*

Proof. Fix H and α such that $H < 1/\alpha$. It suffices to show that $\{L_t^H : t \ge 0\}$ is unbounded on every finite interval, i.e., for each $T > 0$, $\sup_{t \in [0,T]} |L_t^H| = \infty$ a.s. Now, fix $T > 0$. By the equality of finite dimensional distributions proved in Proposition 3.4.1, it is enough to consider $\{Z_t : t \in [0, T]\}$ defined by (3.4.2), instead of $\{L_t^H : t \in [0, T]\}$. For each $t > 0$, the kernel $K_{H,\alpha}(t, x)$ explodes at $x = t$. This implies that Z_t also explodes at $t = T_1, T_2, \cdots$. Therefore, we get $\sup_{t \in [0,T]} |Z_t| = \infty$ a.s. The proof is complete.

Remark 3.4.3. Notice that the kernel $K_{H,\alpha}(t, x)$ also explodes at $x = 0$. (This is even so when $H > 1/\alpha$.) This explosion is, however, irrelevant to the unboundedness of sample paths because $T_i \ne 0$ a.s. for every i. Unfortunately, the preceding theorem implies that fTSm (and its series representation) with short-range dependence is of little practical use.

On the other hand, fTSm with long-range dependence have a better properties. In particular, they are Hölder continuous.

Proposition 3.4.4. *If $H \in (1/\alpha, 1/\alpha + 1/2)$, there exists a continuous modification $\{\widehat{L}_t^H : t \geq 0\}$ of $\{L_t^H : t \geq 0\}$, which is locally Hölder continuous of index γ for every $\gamma \in (0, H - 1/\alpha)$, i.e., for fixed $T > 0$ and for $s, t \in [0, T]$,*

$$\lim_{h \to 0} \sup_{0 < t-s < h} \frac{|\widehat{L}_t^H - \widehat{L}_s^H|}{|t-s|^\gamma} \leq C \quad a.s., \tag{3.4.3}$$

where C is some positive constant.

Proof. By Corollary 3.2.2, we have $E[|L_t^H - L_s^H|^2] = |t-s|^{2G} E[(X_1^{TS})^2]$. If $H > 1/\alpha$, then $2G > 1$, and thus the Kolmogorov-Čentsov Theorem (see, for example, Theorem 3.23 of Kallenberg [20]) directly applies.

Moreover, we can derive some variation properties. Let us first define the notion of the p-variation (process).

Definition 3.4.5. *The p-variation (process) of a stochastic process $\{X_t : t \in [0, T]\}$ is defined by*

$$[X]_t^{(p)} := \sup_{\tau} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^p, \tag{3.4.4}$$

where τ is the collection of all finite partitions $0 = t_0 < t_1 < \cdots < t_n = T$.

Clearly, 1- and 2-variations coincide with the total variation and the quadratic variation, respectively.

Theorem 3.4.6. *$\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ is of zero p-variation for $p \geq 2$ when $H \in (1/\alpha, 1/\alpha + 1/2)$.*

Proof. By the Markov inequality, for each $\epsilon > 0$,

$$\begin{aligned} P\left(\sum_{n=0}^{N-1} |L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T}|^2 > \epsilon\right) &< \epsilon^{-1} E\left[\sum_{n=0}^{N-1} |L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T}|^2\right] \\ &= \epsilon^{-1} (N/T)^{2(1/\alpha - H)} \quad \to 0 \end{aligned}$$

as $N \to \infty$. This implies that $\sum_{n=0}^{N-1} |L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T}|^2 \xrightarrow{p} 0$ as $N \to \infty$. Hence, there exists a subsequence $\{N_k\}_{k\geq 1}$ such that

$$\sum_{n=0}^{N_k-1} |L^H_{\frac{n+1}{N_k}T} - L^H_{\frac{n}{N_k}T}|^2 \to 0 \qquad a.s.$$

as $k \to \infty$, which proves the case $p = 2$. For $p > 2$, we have that

$$\sum_{n=0}^{N-1} |L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T}|^p \leq \max_{0\leq k\leq N-1} |L^H_{\frac{k+1}{N}T} - L^H_{\frac{k}{N}T}|^{p-2} \sum_{n=0}^{N-1} |L^H_{\frac{n+1}{N}T} - L^H_{\frac{n}{N}T}|^2.$$

Since $p - 2 > 0$, the almost sure finiteness of the maximum term follows from the Hölder continuity proven in Theorem 3.4.4. Since the summation term converges to 0 a.s. along a subsequence $\{N_k\}$ and such a subsequence forces the entire right-hand side to converges to 0. The proof is complete.

Proposition 3.4.7. *If $H \in (1/\alpha, 1)$ and $\alpha \in (1, 2)$, then*

$$\lim_{\epsilon \downarrow 0} \sup_{t\in[t_0,t_0+\epsilon]} \left| \frac{L^H_t - L^H_{t_0}}{t - t_0} \right| = \infty \quad a.s.,$$

for each $t_0 \geq 0$.

Proof. We will basically follow the setting of Proposition 3.2 of Houdré and Villa [17]. Define

$$A^{(m)}_n := \left\{ \omega \in \Omega : \sup_{t\in[t_0,t_0+1/n]} \frac{|L^H_t - L^H_{t_0}|}{t - t_0} > m \right\},$$

and $A^{(m)} := \cap_{n=1}^{\infty} A^{(m)}_n$. Clearly, $A^{(m)}_n \subseteq A^{(m)}_{n+1}$ and $A^{(m)} \subseteq A^{(m+1)}$. Hence, it suffices to show that $P(\cap_{m=1}^{\infty} A^{(m)}) = 1$. Since

$$P(\cap_{m=1}^{\infty} A^{(m)}) = \lim_{m\to\infty} P(A^{(m)}) = \lim_{m\to\infty} \lim_{n\to\infty} P(A^{(m)}_n)$$

and $P(A^{(m)}_n) \geq P\left(|L^H_{t_0+1/n} - L^H_{t_0}| > m/n\right)$, we will show

$$\lim_{n\to\infty} P(|L^H_{t_0+1/n} - L^H_{t_0}| \leq m/n) = 0.$$

Now, observe that

$$P(|L^H_{t_0+1/n} - L^H_{t_0}| \leq m/n) = P(n^H |L^H_{t_0+1/n} - L^H_{t_0}| \leq n^{H-1} m).$$

By Theorem 3.3.4, $n^H|L^H_{t_0+1/n} - L^H_{t_0}| \xrightarrow{d} L^{H,\alpha}_{t_0}$ as $n \to \infty$ where $\{L^{H,\alpha}_t : t \geq 0\}$ is a fSm corresponding to $\{L^H_t : t \geq 0\}$. This shows that $n^H|L^H_{t_0+1/n} - L^H_{t_0}|$ is a nondegenerate random variable for every $n > 0$. On the other hand, $n^{H-1}m \to 0$ as $n \to \infty$. The proof is complete.

Proposition 3.4.7 reads that fTSm with $H \in (1/\alpha, 1)$ and $\alpha \in (1, 2)$ is of infinite variation. It is, unfortunately, not clear whether or not this is so when $H \in [1, 1/\alpha + 1/2)$ and $\alpha \in (1, 2)$, or when $H \in (1/\alpha, 1/\alpha + 1/2)$ and $\alpha \in (0, 1)$. The following is a direct consequence of this fact and Proposition 3.4.6.

Theorem 3.4.8. *$\{L^H_t : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ with $H \in (1/\alpha, 1)$ and $\alpha \in (1, 2)$ is not semimartingale.*

Proof. We refer to Lin [26]. For completeness, we give a sketch. Assume that $\{L^H_t : t \geq 0\}$ is a semimartingale. Then, $\{L^H_t : t \geq 0\}$ admits the decomposition $L^H_t = M_t + A_t$, where M is a local martingale and A is of finite variation with $M_0 = A_0 = 0$ a.s. Since $\{L^H_t : t \geq 0\}$ has zero quadratic variation by Proposition 3.4.6,

$$0 = [L^H, L^H]_t = [M, M]_t + 2[M, A]_t + [A, A]_t = [M, M]_t.$$

By the Burkholder-Gundy-Davis inequality, $[M, M]_t = 0$ implies that $M_t \equiv 0$ and thus that $L^H_t = A_t$. However, this contradicts Proposition 3.4.7. The proof is complete.

In Figure 3.2, we give typical sample paths of fTSm. The inner measures ρ_1 and ρ_2 are given by $\rho_1(dx) = \delta_{-1.0}(dx) + \delta_{1.0}(dx)$ and $\rho_2(dx) = 0.5^{-\alpha}\delta_{-0.5}(dx) + \delta_{1.0}(dx)$. Clearly, ρ_1 is symmetric, and ρ_2 is asymmetric. We use ρ_1 and ρ_2 throughout this section. For a better comparison, we also draw their background driving tempered stable processes.

Since fTSm with $H \in (1/\alpha, 1)$ is not a semimartingale, the classical stochastic integration with respect to it is not well defined. However, a slight modification of

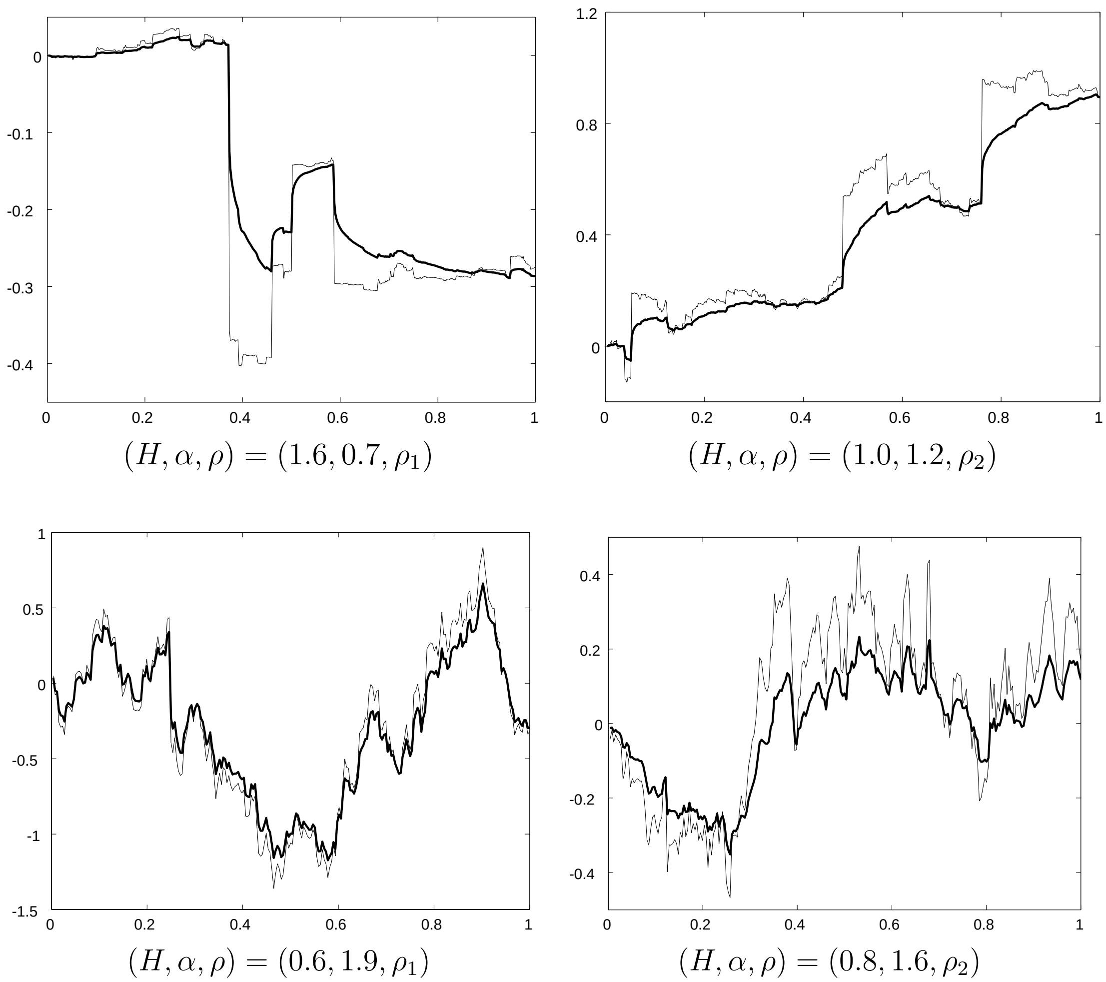


Figure 3.2: fTSm (thick line) and TS (thin line) generated via the series representation

the kernel $K_{H,\alpha}$ induces a semimartingale, which can be arbitrarily close to fTSm. For $H \in (1/\alpha, 1/\alpha + 1/2)$, the kernel $K_{H,\alpha}(\cdot, \cdot)$ can also be written as

$$K_{H,\alpha}(t, s) = c_{H,\alpha}(H - 1/\alpha)s^{1/\alpha - H} \int_s^t (u - s)^{H-1/\alpha-1} u^{H-1/\alpha} du.$$

Let ∂_i denote the partial derivative with respect to i-th argument. Observe that

$$\partial_1 K_{H,\alpha}(t, s) = c_{H,\alpha}(H - 1/\alpha)(t - s)^{H-1/\alpha-1} \left(\frac{t}{s}\right)^{H-1/\alpha},$$

and

$$K_{H,\alpha}(t, s) = \int_s^t \partial_1 K_{H,\alpha}(u, s) du.$$

Therefore,

$$L_t^H = \int_0^t K_{H,\alpha}(t,s)dX_s^{TS} = \int_0^t \left(\int_s^t \partial_1 K_{H,\alpha}(u,s)du \right) dX_s^{TS}. \tag{3.4.5}$$

If the two integrals could be interchanged, then fTSm would be of finite variation, i.e., it would be a semimartingale; we have seen in Proposition 3.4.7 that this is not the case. On the other hand, the integrability condition of the stochastic Fubini's theorem (Theorem 46 (pp.160) of Protter [38]) can be achieved by slightly modifying the kernel $K_{H,\alpha}(t,s)$. Set

$$K_{H,\alpha}^n(t,s) := c_{H,\alpha}(H-1/\alpha)s^{1/\alpha-H} \int_s^t \left(u+\frac{1}{n}-s\right)^{H-1/\alpha-1} u^{H-1/\alpha}du, \tag{3.4.6}$$

where $s \in [0,t]$, $n \in \mathbb{N}$. Then,

$$\partial_1 K_{H,\alpha}^n(t,s) = c_{H,\alpha}(H-1/\alpha)\left(t+\frac{1}{n}-s\right)^{H-1/\alpha-1}\left(\frac{t}{s}\right)^{H-1/\alpha}.$$

The integrability condition is then satisfied; for every $u \in [0,t]$,

$$\begin{aligned}
&\int_0^u \left(\partial_1 K_{H,\alpha}^n(u,s)\right)^2 ds \\
&\qquad = c_{H,\alpha}^2 (H-1/\alpha)^2 \int_0^u \left(u+\frac{1}{n}-s\right)^{2(H-1/\alpha-1)}\left(\frac{u}{s}\right)^{2(H-1/\alpha)} ds \\
&\qquad \le c_{H,\alpha}^2 (H-1/\alpha)^2 (1-2(H-1/\alpha))^{-1} n^{-2(H-1/\alpha-1)} u < \infty,
\end{aligned}$$

and thus the stochastic Fubini's theorem applies. Therefore, we get

$$\int_0^t K_{H,\alpha}^n(t,s)dX_s^{TS} = \int_0^t \left(\int_0^u \partial_1 K_{H,\alpha}^n(u,s)dX_s^{TS}\right)du,$$

which is indeed of finite variation. It would be more interesting if we could further modify it to an infinite variation semimartingale, especially for financial modeling. This can be done as follows. For $\epsilon > 0$, set $K_{H,\alpha}^{n,\epsilon}(t,s) := K_{H,\alpha}^n(t,s) + \epsilon$. Since $\partial_1 K_{H,\alpha}^{n,\epsilon}(u,s) = \partial_1 K_{H,\alpha}^n(u,s)$, the stochastic Fubini's theorem applies and thus

$$\begin{aligned}
\int_0^t K_{H,\alpha}^{n,\epsilon}(t,s)dX_s^{TS} &= \int_0^t (\epsilon + K_{H,\alpha}^n(t,s))dX_s^{TS} \\
&= \epsilon X_t^{TS} + \int_0^t \left(\int_0^u \partial_1 K_{H,\alpha}^n(u,s)dX_s^{TS}\right)du.
\end{aligned}$$

Clearly, this is the definition of the canonical decomposition of semimartingales, i.e. a martingale plus a finite variation process.

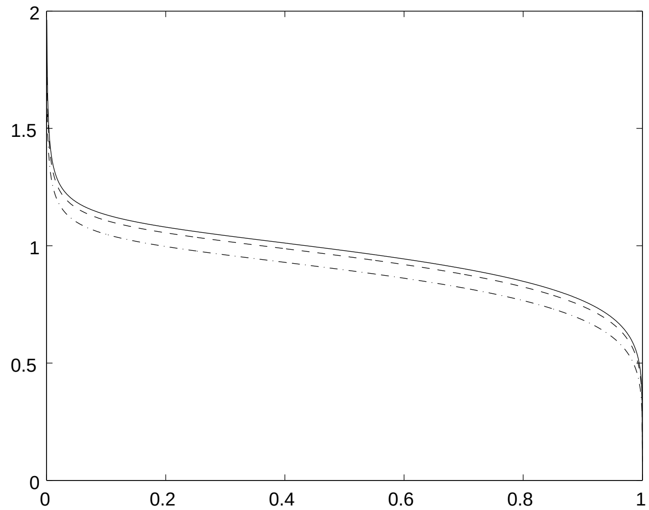


Figure 3.3: Regularized Volterra kernels $K^{n,\epsilon}_{0.7,1.8}(1,\cdot)$; ((—) for $(n,\epsilon)=(\infty,0)$, (- -) for $(10^7, 0.08)$, and (-·-) for $(10^5, 0.12)$)

3.5 Parameter Estimation

Let us first consider the estimation of H. There is a large literature of the estimation of the Hurst parameter via the discrete wavelet transform. All those wavelet-based methods require the strong selfsimilarity and a sample path on the real line $\mathbb{R}$, i.e., $\{X_t,\, t\in\mathbb{R}\}$. In practice, we only have a finite length of sample path $\{X_t,\, t\in[0,T]\}$. The following result is due to Cohen and Istas [10]. It only requires a finite (even very short) length of sample paths and the local selfsimilarity.

Proposition 3.5.1. *Set for $n\in\mathbb{N}$ and $t_0\in[0,t]$,*

$$A_n := \{k\in\mathbb{Z} : |k/2^n - t_0| \le \epsilon_n\}, \quad \epsilon_n > 2^{-n},$$

where $\epsilon_n \to 0$ as $n\to\infty$. Then,

$$-(n(\#A_n))^{-1}\sum_{k\in A_n}\log_2 |L^H_{\frac{k+1}{2^n}} - L^H_{\frac{k}{2^n}}| \to H \quad a.s.$$

as $n \to \infty$.

Proof. Following Cohen and Istas [10], it suffices to show that

(i) there exists a random variable Z such that

$$\frac{L^H_{(1+h)t_0} - L^H_{t_0}}{(ht_0)^H} \xrightarrow{d} Z \quad \textit{as } h \to 0,$$

and that

(ii) for $0 \leq s < t < \infty$, $(\log_2 |L^H_t - L^H_s|)^2$ is uniformly integrable.

First, (i) is immediate from Theorem 3.3.4. For (ii), set

$$f(x) = e^2 x 1_{\{x \leq 1\}} + e^{2\sqrt{x}} 1_{\{x>1\}}.$$

Then clearly,

$$E[f((\log_2 |L^H_t - L^H_s|)^2)] \leq e^2 + E|L^H_t - L^H_s|^2.$$

By the second-order stationary increments,

$$E|L^H_t - L^H_s|^2 = E[(X^{TS}_1)^2](t-s)^G < \infty.$$

Finally, since $f(x)/x \to \infty$ as $x \to \infty$, $(\log_2 |L^H_t - L^H_s|)^2$ is uniformly integrable. The proof is complete.

Next, let us consider the estimation problem of the stability index α of background driving tempered stable processes. Suppose we have obtained $\widehat{H}$ (an estimate of H). If $G(= H - 1/\alpha + 1/2)$ can be estimated, then this gives an estimate $\widehat{\alpha}$ by $(\widehat{H} - \widehat{G} + 1/2)^{-1}$. This can be done by the well-known *aggregated variance method*, making use of second-order selfsimilarity. We give a brief sketch of the method. Suppose we have a finite discrete sample path $\{L^H_i\}^N_{i=1}$, where $N > 0$. Split the time series into m-clusters, i.e., $\{L^H_i\}^m_{i=1}$, $\{L^H_i\}^{2m}_{i=m+1}$, ..., $\{L^H_i\}^N_{i=N-m+1}$, and set

$$Z^{(m)}(k) := m^{-1}(L^H_{km} - L^H_{(k-1)m+1}), \quad k = 1, \ldots, N/m.$$

For each k, we get

$$\mathrm{Var}(Z^{(m)}(k)) = m^{2G-2}(1+1/m)^{2G}\, E[(X_1^{TS})^2] \sim m^{2G-2}\, E[(X_1^{TS})^2]$$

as $m \to \infty$. Compute its sample variance by

$$\widehat{\mathrm{Var}}(Z^{(m)}) = \frac{1}{N/m}\sum_{k=1}^{N/m}(Z^{(m)}(k))^2 - \left(\frac{1}{N/m}\sum_{k=1}^{N/m} Z^{(m)}(k)\right)^2.$$

Repeat this for several m's and the $(\log m) - (\log \widehat{\mathrm{Var}}(Z^{(m)}))$ plot is expected to give the slope of $2G-2$. We assume that m is large so that the dependence among $Z^{(m)}(k)$, $k = 1, \dots N/m$ is negligibly small. Clearly, large N with $N \gg m$ will give a better result.

We thus have obtained estimates of H and α. The rest is the estimation of the inner measure ρ. It is well known that if fBm is defined via the Volterra kernel, then its background driving Wiener process can be recovered as an integral of a resolvent kernel with respect to fBm. The same argument applies to fTSm. Define the operator K by

$$(Kf)(t) := C_{H,\alpha} t^{1/\alpha - H}\left(I_-^{H-1/\alpha} s^{H-1/\alpha} f(s)\right)(t),$$

where $C_{H,\alpha} := (2G(G-1/2)\Gamma(G-1/2)/B(G-1/2, 2-2G))^{1/2}$ and I_-^a is the right-sided Riemann-Liouville fractional integral. Notice that $K_{H,\alpha}(t,s) = K1_{[0,t]}(s)$. For $Kf \in L^2([0,t])$, we define the integral with respect to fTSm by

$$\int_0^t f(s) dL_s^H := \int_0^t (Kf)(s) dX_s^{TS}.$$

The following is useful. (See Sottinen [46] and references therein.)

Lemma 3.5.2. *Let $\{L_t^H : t \geq 0\}$ be fTSm defined by (3.1.3). Then,*

$$X_t^{TS} = \int_0^t K_{H,\alpha}^*(t,s) dL_s^H, \tag{3.5.1}$$

where

$$K_H^*(t,s) = c_{H,\alpha}^*((t/s)^{H-1/\alpha}(t-s)^{1/\alpha-H} \\ -(H-1/\alpha)s^{1/\alpha-H}\int_s^t u^{H-1/\alpha-1/2}(u-s)^{1/\alpha-H}du),$$

with

$$c_{H,\alpha}^* = \frac{\Gamma(1-2H+2/\alpha)}{\Gamma(1/\alpha-H)\sqrt{(2H-2/\alpha+3/2)\Gamma(1/\alpha-H)}}.$$

The integral is in the L^2-sense as well as in the pathwise sense as improper Riemann-Stieltjes integrals. In particular, fTSm $\{L_t^H : t \geq 0\}$ and its background driving tempered stable process $\{X_t^{TS} : t \geq 0\}$ generate the same filtration.

By the inversion formula (3.5.1), we can recover $\{X_t^{TS} : t \in [0,T]\}$ from a single realization of $\{L_t^H : t \in [0,T]\}$. With the recovered $\{X_t^{TS} : t \in [0,T]\}$ and the previously obtained α, the inner measure ρ can also be estimated by a suitable parameterization of ρ. Moreover, Lemma 3.5.2 is applicable to general fractional Lévy motions defined via a Volterra representation with the kernel $K_H(\cdot,\cdot)$.

Remark 3.5.3. The inversion formula (3.5.1) can also be used for prediction of fTSm. Given $\{L_s^H : s \in [0,t]\}$, obtain $\{X_s^{TS} : s \in [0,t]\}$ by (3.5.1). Then, for $T \geq t$,

$$L_T^H = \int_0^T K_{H,\alpha}(T,s)dX_s^{TS} = \int_0^t K_{H,\alpha}(T,s)dX_s^{TS} + \int_t^T K_{H,\alpha}(T,s)dX_s^{TS},$$

where the first term of the right-hand side is deterministic, i.e., the best predictor of L_T^H is given by

$$E[L_T^H|\sigma(L_s^H; s \in [0,t])] = \int_0^t K_{H,\alpha}(T,s)dX_s^{TS},$$

and the residual is $\int_t^T K_{H,\alpha}(T,s)dX_s^{TS}$.

Let us now evaluate the performance of the estimators of H, α, and the inversion formula (3.5.1). We use sample paths of $\{L_t^H : t \geq 0\}$ generated via the series representation.

Let us first check the performance of the estimator for H. We generate $K = 30$ independent realizations of fTSm $\{L_t^H : t \in [0, T]\} \sim fTSm(H, \alpha, \rho)$. For each combination of (H, α, ρ), we obtain estimate; $\{H_k\}_{k=1}^K$. We give estimation results for symmetric ρ_1 in Table 1. $\widehat{H}$ and σ_H are mean and standard deviation of $\{H_k\}_{k=1}^K$, respectively. We also give mean squared error $\text{MSE}_H^2 = K^{-1} \sum_{k=1}^K (H_k - H)^2$.

Table 3.1: Estimation results for H

2^n	2^6	2^7	2^8	2^6	2^7	2^8	2^6	2^7	2^8
	$\alpha = 0.7$								
H	1.5			1.6			1.7		
$\widehat{H}$	1.4801	1.5033	1.4929	1.5836	1.5906	1.6001	1.6944	1.6923	1.6981
σ_H	0.0336	0.0291	0.0182	0.0392	0.0266	0.0190	0.0499	0.0312	0.0139
MSE_H	0.0018	0.0009	0.0003	0.0039	0.0012	0.0002	0.0020	0.0005	0.0003
	$\alpha = 1.0$								
H	1.1			1.2			1.3		
$\widehat{H}$	1.0973	1.0990	1.0981	1.1871	1.1930	1.2001	1.2809	1.2936	1.2916
σ_H	0.0422	0.0293	0.0116	0.0391	0.0200	0.0183	0.0410	0.0239	0.0103
MSE_H	0.0021	0.0005	0.0004	0.0030	0.0008	0.0002	0.0024	0.0007	0.0004
	$\alpha = 1.4$								
H	0.8			0.9			1.0		
$\widehat{H}$	0.7784	0.7932	0.7869	0.8917	0.9033	0.9010	0.9700	0.9811	0.9887
σ_H	0.0437	0.0219	0.0124	0.0491	0.0331	0.0108	0.0523	0.0388	0.0198
MSE_H	0.0023	0.0005	0.0003	0.0019	0.0007	0.0003	0.0031	0.0009	0.0008
	$\alpha = 1.7$								
H	0.6			0.7			0.8		
$\widehat{H}$	0.6030	0.6054	0.6094	0.6889	0.7045	0.7069	0.7927	0.7917	0.7944
σ_H	0.0463	0.0134	0.0118	0.0260	0.0227	0.0132	0.0454	0.0284	0.0144
MSE_H	0.0020	0.0002	0.0002	0.0008	0.0005	0.0002	0.0020	0.0008	0.0002
	$\alpha = 1.9$								
H	0.6			0.7			0.8		
$\widehat{H}$	0.6024	0.6091	0.6144	0.6876	0.7018	0.7113	0.7660	0.7970	0.7973
σ_H	0.0342	0.0278	0.0108	0.0467	0.0224	0.0123	0.0587	0.0275	0.0193
MSE_H	0.0011	0.0008	0.0003	0.0022	0.0005	0.0002	0.0044	0.0007	0.0003

The greater α, the better the results. It is seen that the finer grids give smaller standard deviation and also smaller mean square error, while not necessarily better mean. In total, the grids of 2^7, or 2^8, seem fine enough to obtain very precise estimates. We omit results for asymmetric ρ_2 because they look quite similar to those for ρ_1.

Next, we will consider the estimation of α. As discussed earlier, we estimate G first, and then obtain α by $\alpha = (H - G + 1/2)^{-1}$ where H is given, or well estimated. In view of stock prices, we fix the grid of 2^{-8}, that is, approximately the daily price data set, and compare the performances of $T = 3.0$ (i.e., 3 years), $T = 4.0$, and $T = 5.0$. We give the estimation results for G for symmetric ρ_1 in Table 2.

Table 3.2: Estimation results for G

G	0.6			0.7			0.8		
T	3.0	4.0	5.0	3.0	4.0	5.0	3.0	4.0	5.0
	$\alpha = 0.7$								
$\widehat{G}$	0.5878	0.5762	0.5410	0.6555	0.6477	0.6479	0.7873	0.7720	0.7878
σ_G	0.1085	0.0982	0.1101	0.1170	0.0877	0.1014	0.1720	0.1730	0.1524
MSE_G	0.0957	0.0814	0.1190	0.1073	0.0972	0.1049	0.1547	0.0965	0.0970
	$\alpha = 1.0$								
$\widehat{G}$	0.5530	0.5924	0.5769	0.6840	0.7036	0.6590	0.7845	0.7903	0.7549
σ_G	0.0993	0.0798	0.0763	0.0936	0.0751	0.0943	0.0987	0.0830	0.1016
MSE_G	0.1847	0.1190	0.1241	0.1720	0.1730	0.1524	0.1313	0.1060	0.1097
	$\alpha = 1.4$								
$\widehat{G}$	0.6223	0.6196	0.5912	0.6730	0.7001	0.6878	0.7673	0.8035	0.7744
σ_G	0.0883	0.0510	0.0744	0.1145	0.0796	0.0892	0.1055	0.0909	0.0813
MSE_G	0.0905	0.0519	0.0616	0.1515	0.0782	0.0881	0.1313	0.1730	0.1014
	$\alpha = 1.7$								
$\widehat{G}$	0.5692	0.6012	0.5923	0.6243	0.7012	0.6899	0.7848	0.7991	0.7903
σ_G	0.1032	0.0628	0.0752	0.1078	0.0721	0.0802	0.1384	0.0616	0.0841
MSE_G	0.1087	0.0643	0.0792	0.1020	0.0626	0.0824	0.1286	0.0706	0.0756
	$\alpha = 1.9$								
$\widehat{G}$	0.5861	0.6012	0.5876	0.6590	0.6780	0.6332	0.7873	0.7987	0.7506
σ_G	0.0863	0.0712	0.0890	0.0913	0.0748	0.0813	0.0731	0.0686	0.0733
MSE_G	0.0762	0.0673	0.0710	0.0901	0.0781	0.0798	0.0710	0.0690	0.0753

The mean $\widehat{G}$, the standard deviation σ_G and the mean square error MSE_G are defined in the same manner as in the case of the estimation of H. We observe that the greater α, which gives greater G, the better the accuracy. The length of $T = 4.0$ should be taken for reliable results.

3.6 Concluding Remarks

We have defined and studied the class of fractional tempered stable motions. Most of the nice properties of tempered stable processes are inherited, along with the additional feature of the long-range dependence. We have seen, on the other hand, that fractional tempered stable motions should not be used as an integrator of stochastic integrals due to their non-semimartingale sample paths.

Benassi et al.[5] study the class of "moving-average" fractional Lévy motions, which is defined via the moving-average kernel, $(t-s)_+^{H-1/\alpha} - (s)_+^{H-1/\alpha}$, on the whole real line $\mathbb{R}$. By comparison, our definition seems to be of more practical use in two aspects. First, the stochastic integral of our definition is taken only on the half real line $[0, \infty)$, because of the domain of the Volterra kernel. Second, the Volterra kernel has a nice inversion form, which enables us to perform the prediction as mentioned in Remark 3.5.3. From an empirical point of view, the prediction may be an interesting topic for future research.

Chapter 4

Layering and Mixing Stable Processes

The class of stable processes is among the simplest classes of purely non-Gaussian Lévy processes. The scaling property induced by the simple structure of Lévy measure is a main attraction. Stable processes have been thoroughly studied by many authors and have been used in several fields, such as statistical physics, queueing theory, mathematical finance. Chapter 3 of Sato [44] and the book of Samorodnitsky and Taqqu [45] are referred for basic facts on stable distributions and processes.

In this chapter, we introduce two generalizations of stable processes; *layered stable processes* and *mixed stable processes*. Roughly speaking, a mixed stable process is a mixture of stable processes with a range of stability indices, while a layered stable process jumps in small sizes and in large size by different stability indices. We further extend them to the classes of *tempered layered stable* processes and *tempered mixed stable* processes. In particular, layered stable processes exhibit local- and global-spatiotemporal fractalities while tempered layered stable processes possess local spatiotemporal fractality and global aggregational Gaussianity. For all these classes, we derive series representations, which give further insight into their structures.

4.1 Layered Stable Processes

Definition 4.1.1. *A probability measure μ on $\mathbb{R}^d$ is called layered stable if it is infinitely divisible with no Gaussian component and with Lévy measure $\nu_{lS}^{\alpha,\beta}$ on $\mathbb{R}_0^d$ of the form*

$$\nu_{lS}^{\alpha,\beta}(B) = \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_B(r\xi)(r^{-\alpha}1_{(0,1)}(r) + r^{-\beta}1_{[1,\infty)}(r))\frac{dr}{r}, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.1.1}$$

where $(\alpha, \beta) \in (0, 2) \times (0, 2)$ and λ is a positive finite measure on S^{d-1}.

Clearly, $\nu_{lS}^{\alpha,\beta}$ is well defined as a Lévy measure because

$$\int_{\mathbb{R}_0^d} (\|z\|^2 \wedge 1)\nu_{lS}(dz) = \lambda(S^{d-1}) \left(\frac{1}{2-\alpha} + \frac{1}{\beta} \right).$$

Notice that the class of layered stable distributions contains the class of stable distributions. This is clear by setting $\alpha = \beta$.

We say that a probability measure μ on $\mathbb{R}^d$ is called of class L_0, or selfdecomposable if for any $b > 1$, there exists a probability measure ϱ_b on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. Moreover, the classes $L_m(\mathbb{R}^d)$, $m = 1, 2, \ldots$, are defined recursively as follows; $\mu \in L_m$ if and only if for every $b > 1$ there exists $\varrho_b \in L_{m-1}$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. Clearly, $L_0 \supset L_1 \supset L_2 \supset \cdots$. Define $h_\xi(u) = k_\xi(e^{-u})$. Let us call $h_\xi(u)$ the h-function of μ. It is known that an infinitely divisible distribution μ on $\mathbb{R}^d$ is in $L_0(\mathbb{R}^d)$ if and only if its Lévy measure ν is of the following form;

$$\nu(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi)k_\xi(r)\frac{dr}{r}, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.1.2}$$

with a finite measure σ on S^{d-1} and a nonnegative function $k_\xi(r)$ measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. It is also known that $\mu \in L_0$ is in L_m if and only if $h_\xi(u) \in C^{m-1}$ and $h^{(j)} \geq 0$ for $j = 0, 1, \ldots, m-1$. It follows from the structure of $\nu_{lS}^{\alpha,\beta}$ that layered stable distributions are in $L_0(\mathbb{R}^d)$. The h-function of $\nu_{lS}^{\alpha,\beta}$ is given by

$$h_\xi(u) = e^{\alpha u}1_{(0,\infty)}(u) + e^{\beta u}1_{(-\infty,0]}(u).$$

If $\alpha \neq \beta$, then h is in C^0 but not C^1. Since $h \geq 0$, layered stable distributions are in L_1 but not in L_2.

Proposition 4.1.2. *Let μ be a layered stable distribution with Lévy measure (4.1.1). Then, μ has moment properties as β-stable distributions. In particular,*

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) \begin{cases} < \infty, & p \in (0, \beta), \\ = \infty, & p \in [\beta, \infty). \end{cases}$$

Proof. Clearly, the restriction $[\nu_{lS}^{\alpha,\beta}]_{\{\|z\|>1\}}$ is equivalent to $[\nu^{\beta,\beta}]_{\{\|z\|>1\}}$. Hence, by Theorem 1.2.2, μ possesses the same moment properties as β-stable distributions.

Definition 4.1.3. *A Lévy process $\{X_t^{lS} : t \geq 0\}$ in $\mathbb{R}^d$ is called a layered stable process if it is generated by $(\gamma, 0, \nu_{lS}^{\alpha,\beta})$ for some $\gamma \in \mathbb{R}^d$. Moreover, define an a-stable Lévy process $\{X_t^{(a)} : t \geq 0\}$ by the characteristic function of $X_1^{(a)}$ as follows,*

$$E[e^{i\langle y, X_1^{(a)}\rangle}] = \begin{cases} \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1)\nu_{lS}^{a,a}(dz)\right], & if\ a \in (0,1), \\ \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle 1_{\{\|z\|<1\}}(z))\nu_{lS}^{1,1}(dz)\right], & if\ a = 1, \\ \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle)\nu_{lS}^{a,a}(dz)\right], & if\ a \in (1,2). \end{cases} \tag{4.1.3}$$

The following theorem tells us that a layered stable process with $\alpha \leq \beta$ behaves like a stable process with stability index α in a short time, while it behaves like a stable process with stability index β in a long time.

Theorem 4.1.4. *Let $\{X_t^{lS} : t \geq 0\}$ be a layered stable process in $\mathbb{R}^d$ having the characteristic function*

$$E[e^{i\langle y, X_t^{lS}\rangle}] = \exp\left[t \int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle 1_{\{\|z\|<1\}}(z))\nu_{lS}^{\alpha,\beta}(dz)\right],$$

with $\alpha \leq \beta$. Also let $\{X_t^{(a)} : t \geq 0\}$ be an a-stable process defined by (4.1.3). Then, *(i) Short-time behavior;*

$$\{h^{-1/\alpha}(X_{ht}^{lS} + htb_\alpha) : t \geq 0\} \xrightarrow{d} \{X_t^{(\alpha)} : t \geq 0\} \quad as\ h \to 0, \tag{4.1.4}$$

and,

(ii) Long-time behavior;

$$\{h^{-1/\beta}(X_{ht}^{lS} + htb_\beta) : t \geq 0\} \xrightarrow{d} \{X_t^{(\beta)} : t \geq 0\} \quad as\ h \to \infty, \tag{4.1.5}$$

where both convergences hold for finite dimensional distributions, and

$$b_a = \begin{cases} (1-\alpha)^{-1} \int_{S^{d-1}} \xi \lambda(d\xi), & if\ a < 1, \\ 0, & if\ a = 1, \\ (1-\beta)^{-1} \int_{S^{d-1}} \xi \lambda(d\xi), & if\ a > 1. \end{cases}$$

Proof. Since a layered stable processes is a Lévy process, it suffices to show the weak convergence of their marginal at time 1. We will follow Theorem 15.14 of Kallenberg [20]. For a Lévy measure ν, let

$$B_a(\nu) := \begin{cases} \int_{\|z\| \leq 1} z\nu(dz), & \text{if } a < 1, \\ 0, & \text{if } a = 1, \\ \int_{\|z\| > 1} z\nu(dz), & \text{if } a > 1. \end{cases}$$

Let us first prove (i). Clearly, $b_\alpha = B_\alpha(\nu_{lS}^{\alpha,\beta})$. Then, for each $h > 0$, the random variable $h^{-1/\alpha}(X_h^{lS} + hB_\alpha(\nu_{lS}^{\alpha,\beta}))$ is infinitely divisible with the generating triplet

$$\left(B_\alpha(h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})), 0, h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})\right).$$

We need to show that for each $\kappa > 0$,

$$h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta}) \xrightarrow{v} \nu_{lS}^{\alpha,\alpha}, \tag{4.1.6}$$

$$\int_{\|z\| \leq \kappa} zz' h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz) \to \int_{\|z\| \leq \kappa} zz' \nu_{lS}^{\alpha,\alpha}(dz), \tag{4.1.7}$$

$$B_\alpha(h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})) - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz) \to B_\alpha(\nu_{lS}^{\alpha,\alpha}) - \int_{\kappa < \|z\| \leq 1} z\nu_{lS}^{\alpha,\alpha}(dz), \tag{4.1.8}$$

as $h \to 0$, where $\xrightarrow{v}$ denotes a convergence in the vague topology. For (4.1.6), it suffices to show that

$$\lim_{h\to 0}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz) = \int_{\mathbb{R}_0^d} f(z)\nu_{lS}^{\alpha,\alpha}(dz),$$

where f is a bounded continuous function from $\mathbb{R}_0^d$ to $\mathbb{R}$ vanishing on a neighborhood of the origin. For such a function f, there exists an $\epsilon_1 > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz)\\ &\qquad= \int_{S^{d-1}}\lambda(d\xi)\int_{\epsilon_1}^{\infty} f(r\xi)\frac{dr}{r^{\alpha+1}1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}}r^{\beta+1}1_{[h^{-1/\alpha},\infty)}(r)}\end{aligned}$$

and then the right hand side converges as $h \to 0$ to

$$\int_{S^{d-1}}\lambda(d\xi)\int_{\epsilon_1}^{\infty} f(r\xi)\frac{dr}{r^{\alpha+1}} = \int_{\mathbb{R}_0^d} f(z)\nu_{lS}^{\alpha,\alpha}(dz),$$

provided the limit and the integral can be interchanged. It remains to justify the passage to the limit. Let C be a constant such that $|f| \leq C$. Then, we have for each $\xi \in S^{d-1}$ and $h < \epsilon_1^{-\alpha}$,

$$\begin{aligned}&\left|\int_{\epsilon_1}^{\infty} f(r\xi)\frac{dr}{r^{\alpha+1}1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}}r^{\beta+1}1_{[h^{-1/\alpha},\infty)}(r)}\right|\\ &\qquad\qquad\leq C\left(\int_{\epsilon_1}^{h^{-1/\alpha}}\frac{dr}{r^{\alpha+1}} + \int_{h^{-1/\alpha}}^{\infty}\frac{dr}{h^{\frac{\beta-\alpha}{\alpha}}r^{\beta+1}}\right)\\ &\qquad\qquad= C\left(h\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) + \frac{\epsilon_1^{-\alpha}}{\alpha}\right) < C\frac{\epsilon_1^{-\alpha}}{\alpha} < \infty,\end{aligned}$$

which proves (4.1.6). Now we will prove (4.1.7) and (4.1.8). For (4.1.7), we can easily verify that for each $h < \kappa^{-\alpha}$,

$$\int_{\|z\|\leq\kappa} zz^T h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz) = \frac{\kappa^{2-\alpha}}{2-\alpha}\int_{S^{d-1}}\xi\xi^T\lambda(d\xi) = \int_{\|z\|\leq\kappa} zz^T\nu^{\alpha,\alpha}(dz),$$

and thus (4.1.7) is satisfied. Indeed, we do not have to take a limit for (4.1.8), either. In fact, if $\alpha \in (0,1)$, for each $h < \kappa^{-\alpha}$, we have

$$\begin{aligned}B_\alpha(h(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})) - \int_{\kappa<\|z\|\leq 1} zh(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz) &= \int_{\|z\|\leq\kappa} zh(T_{h^{-1/\alpha}}\nu_{lS}^{\alpha,\beta})(dz)\\ = \frac{k^{1-\alpha}}{1-\alpha}\int_{S^{d-1}}\xi\lambda(d\xi) = B_\alpha(\nu_{lS}^{\alpha,\beta}) - \int_{\kappa<\|z\|\leq 1} z\nu_{lS}^{\alpha,\beta}(dz).&\end{aligned}$$

The cases $\alpha = 1$ and $\alpha \in (1,2)$ are just similar.

Now, (ii) can be proved similarly. Clearly, $b_\beta = B_\beta(\nu_{lS}^{\alpha,\beta})$, and for each $h > 0$, the law of $h^{-1/\beta}(X_h^{lS} + hB_\beta(\nu_{lS}^{\alpha,\beta}))$ is infinitely divisible with the generating triplet

$$\left(B_\beta(h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})), 0, h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})\right).$$

We need to show that for each $\kappa > 0$,

$$h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta}) \xrightarrow{v} \nu_{lS}^{\beta,\beta}, \tag{4.1.9}$$

$$\int_{\|z\|\le\kappa} zz^T h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz) \to \int_{\|z\|\le\kappa} zz^T \nu_{lS}^{\beta,\beta}(dz), \tag{4.1.10}$$

$$B_\beta(h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})) - \int_{\kappa<\|z\|\le 1} zh(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz) \to B_\beta(\nu_{lS}^{\beta,\beta}) - \int_{\kappa<\|z\|\le 1} z\nu_{lS}^{\beta,\beta}(dz), \tag{4.1.11}$$

as $h \to \infty$. We omit proofs of (4.1.10) and (4.1.11) because they are just similar to those of (4.1.7) and (4.1.8). For (4.1.9), we will show that

$$\lim_{h\to\infty} \int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz) = \int_{\mathbb{R}_0^d} f(z)\nu_{lS}^{\beta,\beta}(dz),$$

where f is a bounded continuous function from $\mathbb{R}_0^d$ to $\mathbb{R}$ vanishing on a neighborhood of the origin. For such f, there exists an $\epsilon_2 > 0$ such that

$$\begin{aligned}
&\lim_{h\to\infty} \int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz) \\
&\quad= \lim_{h\to\infty} \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_2}^\infty f(r\xi) \frac{dr}{h^{\frac{\alpha-\beta}{\beta}} r^{\alpha+1} 1_{(0,h^{-1/\beta})}(r) + r^{\beta+1} 1_{[h^{-1/\beta},\infty)}(r)} \\
&\quad= \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_2}^\infty f(r\xi) \frac{dr}{r^{\beta+1}} = \int_{\mathbb{R}_0^d} f(z)\nu_{lS}^{\beta,\beta}(dz),
\end{aligned}$$

where the passage to the limit is justified by, for each $\xi \in S^{d-1}$ and $h > \epsilon_2^\beta$,

$$\begin{aligned}
&\left| \int_{\epsilon_1}^\infty f(r\xi) \frac{dr}{r^{\alpha+1} 1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}} r^{\beta+1} 1_{[h^{-1/\alpha},\infty)}(r)} \right| \\
&\qquad\le C \int_{\epsilon_2}^\infty \frac{dr}{r^{\beta+1}} = C\epsilon_2^{-\beta}/\beta < \infty,
\end{aligned}$$

which proves (ii). The proof is complete.

The short-time and long-time behaviors capture, for example, the dynamics of asset prices; they consists of big jumps in a short-time framework, while they look almost like continuous paths in a long-time framework. It is worth noting that similar results do not hold if $\alpha > \beta$.

Let us derive a series representation of a layered stable process with $\alpha < \beta$.

Theorem 4.1.5. *Let $\{X_t^{lS} : t \in [0,1]\}$ be a layered stable process in $\mathbb{R}^d$ with $\alpha < \beta$ having the characteristic function*

$$E[e^{i\langle y, X_t^{lS}\rangle}] = \begin{cases} \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1)\nu_{lS}^{\alpha,\beta}(dz)\right], & if \ \alpha \in (0,1), \\ \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle)\nu_{lS}^{\alpha,\beta}(dz)\right], & if \ \alpha \in [1,2). \end{cases} \tag{4.1.12}$$

Then,

$$\{X_t^{lS} : t \in [0,T]\} \stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[\left(\left(\frac{\beta \Gamma_i}{\lambda(S^{d-1})T} \right)^{-1/\beta} 1_{(0,\lambda(S^{d-1})T/\beta]}(\Gamma_i) \right.\right.\right.$$
$$\left. + \left(\frac{\alpha \Gamma_i}{\lambda(S^{d-1})T} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} 1_{(\lambda(S^{d-1})T/\beta, \infty)}(\Gamma_i) \right) V_i 1(T_i \leq t)$$
$$\left.\left. - z_0 c_i \frac{t}{T} \right] : t \in [0,T] \right\},$$

where the equality holds for finite dimensional distributions and the convergence of the sum is almost surely uniformly on $[0,T]$. Here, $\{T_i\}_{i\geq 1}$ is a sequence of iid uniform random variables in $[0,1]$, $\{\Gamma_i\}_{i\geq 1}$ are Poisson arrivals with rate 1, $\{V_i\}_{i\geq 1}$ is a sequence of iid random vectors in S^{d-1} with the common distribution σ defined by

$$\sigma(C) = \lambda(C)/\lambda(S^{d-1}), \quad C \in \mathcal{B}(S^{d-1}),$$

and $z_0 = \int_{S^{d-1}} \xi \lambda(d\xi)/\lambda(S^{d-1})$. Moreover, $\{T_i\}_{i\geq 1}$, $\{\Gamma_i\}_{i\geq 1}$, $\{V_i\}_{i\geq 1}$ are all mutually independent. Finally, $\{c_i\}_{i\geq 1}$ is a sequence of constants given by, if $\alpha \in (0,1)$, $c_i \equiv 0$, *if* $\alpha = 1$,

$$\begin{aligned} c_i &= \left(\frac{\beta}{\lambda(S^{d-1})T}\right)^{-1/\beta} \frac{(i \wedge \lambda(S^{d-1})T/\beta) - ((i-1) \wedge \lambda(S^{d-1})T/\beta)}{1-1/\beta} \\ &\quad + \lambda(S^{d-1})T \Bigg[\ln\left(\frac{i \vee \lambda(S^{d-1})T/\beta}{\lambda(S^{d-1})T} + 1 - \frac{1}{\beta}\right) \\ &\qquad - \ln\left(\frac{(i-1) \vee \lambda(S^{d-1})T/\beta}{\lambda(S^{d-1})T} + 1 - \frac{1}{\beta}\right)\Bigg], \end{aligned}$$

and if $\alpha \in (1,2)$,

$$\begin{aligned} c_i = &\left(\frac{\beta}{\lambda(S^{d-1})T}\right)^{-1/\beta} \frac{(i \wedge \lambda(S^{d-1})T/\beta) - ((i-1) \wedge \lambda(S^{d-1})T/\beta)}{1-1/\beta} \\ &+ \frac{\lambda(S^{d-1})T}{\alpha - 1}\Bigg[\left(\frac{\alpha(i \vee \lambda(S^{d-1})T/\beta)}{\lambda(S^{d-1})T} + 1 - \frac{1}{\beta}\right)^{1-1/\alpha} \\ &\qquad - \left(\frac{\alpha((i-1) \vee \lambda(S^{d-1})T/\beta)}{\lambda(S^{d-1})T} + 1 - \frac{1}{\beta}\right)^{1-1/\alpha}\Bigg]. \end{aligned}$$

Proof. By Theorem 1.4.1 and the LePage's method, when $\alpha \in (0,1)$,

$$\{X_t^{lS} : t \in [0,T]\} \stackrel{d}{=} \left\{\sum_{i=1}^{\infty} \overleftarrow{q}(\Gamma_i/T)V_i 1(T_i \le t) : t \in [0,T]\right\},$$

and when $\alpha \in [1,2)$,

$$\begin{aligned} \{X_t^{lS} : t \in [0,T]\} \stackrel{d}{=} \Bigg\{ \sum_{i=1}^{\infty} \Bigg[&\overleftarrow{q}(\Gamma_i/T)V_i 1(T_i \le t) \\ &- \int_{i-1}^{i} E[\overleftarrow{q}(s/T)V_i 1(T_i \le t)]ds \Bigg] : t \in [0,T] \Bigg\}, \end{aligned}$$

where $\overleftarrow{q} : \mathbb{R}^+ \to (0,\infty)$ is given by

$$\begin{aligned} \overleftarrow{q}(u) &= \inf\left\{x > 0 : \lambda(S^{d-1}) \int_x^{\infty} \frac{dr}{r^{\alpha+1}1_{(0,1)}(r) + r^{\beta+1}1_{[1,\infty)}(r)} < u\right\} \\ &= \left(\frac{\beta u}{\lambda(S^{d-1})}\right)^{-1/\beta} 1_{(0,\lambda(S^{d-1})/\beta]}(u) \\ &\qquad + \left(\frac{\alpha u}{\lambda(S^{d-1})} + 1 - \frac{\alpha}{\beta}\right)^{-1/\alpha} 1_{(\lambda(S^{d-1})/\beta,\infty)}(u). \end{aligned}$$

Finally, by the elementary computation, we can obtain, for each i,

$$\int_{i-1}^{i} E[\overleftarrow{q}(s/T)V_i 1(T_i \le t)]ds = z_0 c_i t/T,$$

which concludes the proof.

The condition $\alpha \le \beta$ is crucial so that the inverse q-function is well defined.

Remark 4.1.6. All jumps with absolute size smaller than 1 are due to the β-stable series $\left(\frac{\beta\Gamma_i}{\lambda(S^{d-1})}\right)^{-1/\beta} V_i$, while bigger jumps come from $\left(\frac{\alpha\Gamma_i}{\lambda(S^{d-1})} + 1 - \frac{\alpha}{\beta}\right)^{-1/\alpha} V_i$, which resembles α-stable jumps. This fact reveals the nature of layering of stable jumps. Moreover, the short-time and long-time behaviors can be derived from the series representation. For simplicity, consider a symmetric layered stable process, i.e., $z_0 = 0$. Observe that

$$\begin{aligned} h^{-1/\alpha} X_h^{lS} &\stackrel{d}{=} \sum_{i=1}^{\infty} \Bigg[h^{-1/\alpha} \left(\frac{\beta\Gamma_i/h}{\lambda(S^{d-1})}\right)^{-1/\beta} 1_{(0,\lambda(S^{d-1})/\beta]}(\Gamma_i/h) \\ &\qquad + \left(h\left(\frac{\alpha\Gamma_i/h}{\lambda(S^{d-1})} + 1 - \frac{\alpha}{\beta}\right)\right)^{-1/\alpha} 1_{(\lambda(S^{d-1})/\beta,\infty)}(\Gamma_i/h)\Bigg] V_i \\ &\to \sum_{i=1}^{\infty} \left(\frac{\alpha\Gamma_i}{\lambda(S^{d-1})}\right)^{-1/\alpha} V_i \quad a.s., \quad as \quad h \to 0, \end{aligned}$$

which shows the short-time behavior. The long-time behavior can be shown similarly.

Remark 4.1.7. We can derive two more forms of series representations when $\alpha < \beta$ via the rejection method. This can be seen by, for $z \in \mathbb{R}_0^d$,

$$\frac{d\nu_{lS}^{\alpha,\beta}}{d\nu_{lS}^{\alpha,\alpha}}(z) = 1_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta} 1_{(1,\infty)}(\|z\|) \le 1,$$

and

$$\frac{d\nu_{lS}^{\alpha,\beta}}{d\nu_{lS}^{\beta,\beta}}(z) = \|z\|^{\beta-\alpha} 1_{(0,1]}(\|z\|) + 1_{(1,\infty)}(\|z\|) \le 1.$$

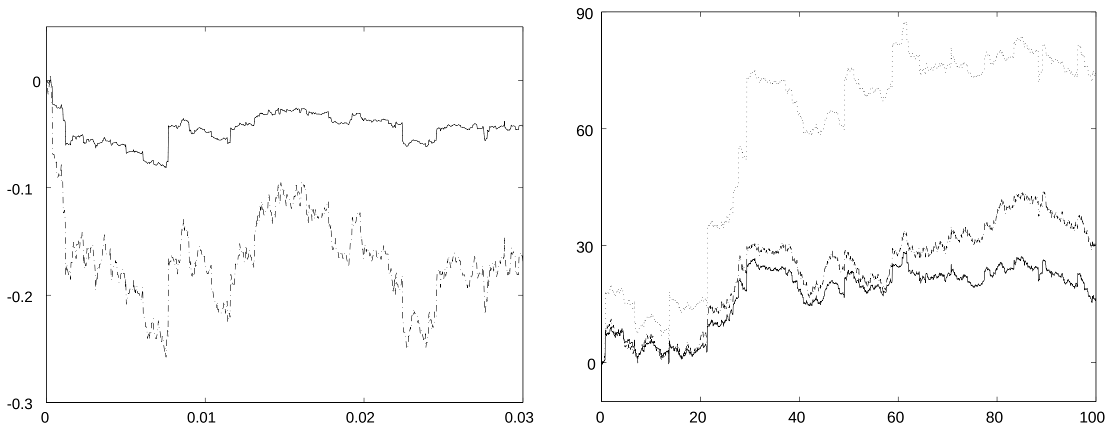
Hence, H-sequences can be given by

$$H_i = \left(\frac{\alpha\Gamma_i}{\lambda(S^{d-1})}\right)^{-1/\alpha} 1\left(\frac{d\nu_{lS}^{\alpha,\beta}}{d\nu_{lS}^{\alpha,\alpha}}((\alpha\Gamma_i/\lambda(S^{d-1}))^{-1/\alpha} V_i) \ge U_i\right),$$

or

$$H_i = \left(\frac{\beta\Gamma_i}{\lambda(S^{d-1})}\right)^{-1/\beta} 1\left(\frac{d\nu_{lS}^{\alpha,\beta}}{d\nu_{lS}^{\beta,\beta}}((\beta\Gamma_i/\lambda(S^{d-1}))^{-1/\beta} V_i) \ge U_i\right),$$

where $\{U_i\}_{i\ge 1}$ is a sequence of iid uniform random variables in $[0,1]$.



$t \in [0, 0.03]$ $t \in [0, 100]$

Figure 4.1: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.3, 1.9)$, 1.3-stable process $(\cdots)$, and 1.9-stable process (-·-)

In Figure 4.1, we give typical sample paths of layered stable processes $(\alpha, \beta) = (1.3, 1.9)$, together with 1.3- and 1.9-stable processes. The short-time and long-time behaviors are apparent. (In the left figure, the layered stable process and the 1.3-stable process are almost indistinguishable.)

Definition 4.1.8. *A probability measure μ on $\mathbb{R}^d$ is called tempered layered stable if it is infinitely divisible with no Gaussian component and with Lévy measure $\nu_{TlS}^{\alpha,\beta}$ on $\mathbb{R}_0^d$ given by*

$$\nu_{TlS}^{\alpha,\beta}(B) = \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{q(r,\xi)}{r^{\alpha+1} 1_{(0,1)}(r) + r^{\beta+1} 1_{[1,\infty)}(r)} dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.1.13}$$

where $(\alpha, \beta) \in (0,2) \times (0,2)$, $q : (0,\infty) \times S^{d-1} \to (0,\infty)$ is a Borel function such that $q(\cdot, \xi)$ is completely monotone with $\lim_{r\to\infty} q(r,\xi) \to 0$ for every $\xi \in S^{d-1}$, and λ is a finite measure on S^{d-1} such that $\int_{S^{d-1}} q(0+,\xi)\lambda(d\xi) < \infty$. Moreover, a Lévy process in $\mathbb{R}^d$ generated by $(b, 0, \nu_{TlS}^{\alpha,\beta})$ for some $b \in \mathbb{R}^d$ is called a tempered layered stable process.

Like tempered stable processes, tempered layered stable processes also possess a short time behavior. Define an a-stable Lévy measure $\nu_{(a)}$ on $\mathbb{R}^d_0$ by

$$\nu_a(B) = \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{q(0+,\xi)}{r^{a+1}} dr, \quad B \in \mathcal{B}(\mathbb{R}^d_0).$$

We denote by $\{X_t^{(a)} : t \geq 0\}$ an a-stable Lévy process having the characteristic function

$$E[e^{i\langle y, X_1^{(a)}\rangle}] = \begin{cases} \exp\left[\int_{\mathbb{R}^d_0} (e^{i\langle y,z\rangle} - 1)\nu_a(dz)\right], & \text{if } a \in (0,1), \\ \exp\left[\int_{\mathbb{R}^d_0} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle 1_{\|z\|\leq 1}(z))\nu_a(dz)\right], & \text{if } a = 1, \\ \exp\left[\int_{\mathbb{R}^d_0} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle)\nu_a(dz)\right], & \text{if } a \in (1,2). \end{cases}$$

Theorem 4.1.9. *Let $\{X_t^{TlS} : t \geq 0\}$ be a tempered layered stable process in $\mathbb{R}^d$ generated by $(0, 0, \nu_{TlS}^{\alpha,\beta})$.*

(i) Short-time behavior; Then,

$$\{h^{-1/\alpha}(X_{ht}^{TlS} - b\,ht) : t \geq 0\} \xrightarrow{d} \{X_t^{(\alpha)} : t \geq 0\} \quad \text{as } h \to 0, \tag{4.1.14}$$

where the convergence holds for finite dimensional distributions and

$$b = \begin{cases} -\int_{S^{d-1}} \int_0^1 \frac{\xi q(r,\xi)}{r^{\alpha+1}} dr \lambda(d\xi), & if\ \alpha \in (0,1), \\ 0, & if\ \alpha = 1, \\ \int_{S^{d-1}} \int_1^\infty \frac{\xi q(r,\xi)}{r^{\beta+1}} dr \lambda(d\xi), & if\ \alpha \in (1,2). \end{cases}$$

(ii) Long-time behavior; Assume

$$\int_{S^{d-1}} \lambda(d\xi) \int_1^\infty \frac{q(r,\xi)}{r^{\beta-1}} dr < \infty. \tag{4.1.15}$$

Then,

$$\{h^{-1/2}(X_{ht}^{TlS} - c\,ht) : t \geq 0\} \xrightarrow{d} \{B_t : t \geq 0\} \quad \text{as } h \to 0, \tag{4.1.16}$$

where the convergence holds for finite dimensional distributions, $\{B_t\}$ is a Brownian motion with the Gaussian covariance matrix A given by

$$A = \int_{\mathbb{R}^d_0} zz^T \, \nu_{TlS}(dz),$$

and

$$c = \int_{S^{d-1}} \xi \lambda(d\xi) \int_1^\infty \frac{q(r,\xi)}{r^\beta} dr.$$

Proof. Let us first prove (i). As in the proof of Theorem 4.1.4, we need to show a vague convergence of Lévy measure, a convergence of the constant component and the Gaussian component. We will only show the vague convergence because the rest are just similar. (Although the vague convergence will also be proved similarly, we need to show the passage to the limit.) For $f : \mathbb{R}^d \to \mathbb{R}$ bounded continuous vanishing on a neighborhood of the origin, there exists an $\epsilon > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/\alpha}} \nu_{TlS}^{\alpha,\beta})(dz) \\
&\quad = \int_{S^{d-1}} \lambda(d\xi) \int_\epsilon^\infty f(r\xi) \frac{q(h^{1/\alpha} r, \xi) dr}{r^{\alpha+1} 1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}} r^{\beta+1} 1_{[h^{-1/\alpha},\infty)}(r)},
\end{aligned}$$

and the last integral converges as $h \to 0$ to

$$\int_{S^{d-1}} q(0+,\xi) \lambda(d\xi) \int_\epsilon^\infty f(r\xi) \frac{dr}{r^{\alpha+1}}.$$

Now we will justify the interchange of the limit with the integral. Let C be a constant such that $|f| \le C < \infty$. Since $\xi \in S^{d-1}$, $q(r,\xi) \le q(0+,\xi)$ for each $\xi \in S^{d-1}$ due to the absolute monotonicity, we have for $h < \epsilon^{-\alpha}$,

$$\begin{aligned}
&\left| \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_1}^\infty f(r\xi) \frac{q(h^{1/\alpha} r, \xi) dr}{r^{\alpha+1} 1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}} r^{\beta+1} 1_{[h^{-1/\alpha},\infty)}(r)} \right| \\
&\quad \le \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_1}^\infty |f(r\xi)| \frac{q(0+,\xi) dr}{r^{\alpha+1} 1_{(0,h^{-1/\alpha})}(r) + h^{\frac{\beta-\alpha}{\alpha}} r^{\beta+1} 1_{[h^{-1/\alpha},\infty)}(r)} \\
&\quad \le C \int_{S^{d-1}} q(0+,\xi) \lambda(d\xi) \left(\int_{\epsilon_1}^{h^{-1/\alpha}} \frac{dr}{r^{\alpha+1}} + \int_{h^{-1/\alpha}}^\infty \frac{dr}{h^{\frac{\beta-\alpha}{\alpha}} r^{\beta+1}} \right) \\
&\quad < C \int_{S^{d-1}} q(0+,\xi) \lambda(d\xi) \frac{\epsilon^{-\alpha}}{\alpha} < \infty,
\end{aligned}$$

which proves (i).

Next, we will prove (ii). First, notice that (4.1.15) ensures that A and c are well defined. In fact,

$$\int_{\mathbb{R}_0^d} \|z\|^2 \nu_{TlS}^{\alpha,\beta}(dz) \le \frac{1}{2-\alpha} \int_{S^{d-1}} q(0+,\xi) \lambda(d\xi) + \int_{S^{d-1}} \lambda(d\xi) \int_1^\infty \frac{q(r,\xi)}{r^{\beta-1}} dr < \infty$$

and

$$\int_{\|z\|>1} \|z\| \nu_{TlS}^{\alpha,\beta}(dz) \leq \int_{\|z\|>1} \|z\|^2 \nu_{TlS}^{\alpha,\beta}(dz) = \int_{S^{d-1}} \lambda(d\xi) \int_1^\infty \frac{q(r,\xi)}{r^{\beta-1}} dr < \infty.$$

Since $q(\cdot,\xi)$ is completely monotone, there exists a measure $\varrho_\xi(\cdot)$ on $S^{d-1} \times (0,\infty)$ such that $q(r,\xi) = \int_{(0,\infty)} e^{-rx} \varrho_\xi(dx)$ (and the measure ϱ_ξ is uniquely determined.) As before, for $f : \mathbb{R}^d \to \mathbb{R}$ bounded continuous vanishing on a neighborhood of the origin, we have for some $\epsilon_1 > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/2}} \nu_{TlS}^{\alpha,\beta})(dz) \\
&\quad = \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_1}^\infty f(r\xi) \frac{q(h^{1/2} r, \xi) dr}{h^{(\alpha-2)/2} r^{\alpha+1} 1_{(0,h^{-1/2}]}(r) + h^{(\beta-2)/2} r^{\beta+1} 1_{(h^{-1/2},\infty)}(r)} \\
&\quad = \int_{S^{d-1}} \lambda(d\xi) \int_{\epsilon_1}^\infty f(r\xi) \frac{\int_{(0,\infty)} e^{-h^{1/2} r x} \varrho_\xi(dx) dr}{h^{(\alpha-2)/2} r^{\alpha+1} 1_{(0,h^{-1/2}]}(r) + h^{(\beta-2)/2} r^{\beta+1} 1_{(h^{-1/2},\infty)}(r)},
\end{aligned}$$

which tends to 0 as $h \to \infty$. The interchange of the limit and the integral is justified by

$$\left| \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/2}} \nu_{TlS}^{\alpha,\beta})(dz) \right| \leq \frac{C \epsilon_2^{-\beta}}{\beta} \int_{S^{d-1}} q(0+,\xi) \lambda(d\xi) < \infty,$$

for $h > \epsilon_2^\beta$. Finally, for every $\kappa > 0$,

$$\int_{\|z\| \leq \kappa} z z^T h(T_{h^{-1/2}} \nu_{TlS})(dz) = \int_{\|z\| \leq h^{1/2}\kappa} z z^T \nu_{TlS}(dz) \to \int_{\mathbb{R}_0^d} z z^T \nu_{TlS}(dz),$$

and $\int_{\|z\|>\kappa} z h(T_{h^{-1/2}} \nu_{TlS})(dz) \to 0$, as $h \to \infty$, which proves (ii).

We state in Remark 4.1.7 that two forms of series representations of layered stable processes can be derived by the rejection method by using the absolute continuity of their Lévy measure with respect to an stable Lévy measure. The same concept applies to tempered layered stable processes.

Theorem 4.1.10. *Let $\{X_t^{TlS} : t \in [0,1]\}$ be a tempered layered stable process in $\mathbb{R}^d$ generated by $(0, 0, \nu_{TlS}^{\alpha,\beta})$ with $\alpha < \beta$. Set $g : \mathbb{R}_0^d \to [0,1]$ by*

$$g(z) = \begin{cases} \frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)} (1_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta} 1_{(1,\infty)}(\|z\|)), & \text{if } q(0+, z/\|z\|) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then,

$$\{X_t^{TlS} : t \in [0,1]\}$$
$$\stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left(\frac{\alpha \Gamma_i}{\lambda_q} \right)^{-1/\alpha} V_i 1 \left(g\left(\left(\frac{\alpha \Gamma_i}{\lambda_q} \right)^{-1/\alpha} V_i \right) \geq U_i \right) 1(T_i \leq t) - c_i t : t \in [0,1] \right\},$$

where the equality is in the sense of finite dimensional distributions, $\{\Gamma_i\}_{i\geq 1}$ *are Poisson arrivals with rate* 1, $\{U_i\}_{i\geq 1}$ *and* $\{T_i\}_{i\geq 1}$ *are sequences of iid uniform random variables in* $[0,1]$, $\{V_i\}_{i\geq 1}$ *is a sequence of iid random vectors in* S^{d-1} *with the common distribution*

$$\sigma(d\xi) = \lambda_q^{-1} q(0+, \xi)\lambda(d\xi), \quad \xi \in S^{d-1},$$

and $\lambda_q = \int_{S^{d-1}} q(0+, \xi)\lambda(d\xi)$. *Finally,* $\{c_i\}_{i\geq 1}$ *is a sequence of constants given by*

$$c_i = \int_{(i-1)\vee \lambda_q/\alpha}^{i \vee \lambda_q/\alpha} \left(\frac{\alpha s}{\lambda_q} \right)^{-1/\alpha} \left(z_0 - \lambda_q^{-1} \int_{S^{d-1}} g((\alpha s/\lambda_q)^{-1/\alpha}\xi)\xi\lambda(d\xi) \right) ds,$$

where $z_0 = \lambda_q^{-1} \int_{S^{d-1}} q(0+, \xi)\xi\lambda(d\xi)$.

Proof. Observe that

$$\frac{d\nu_{TlS}^{\alpha,\beta}}{d\nu_{(\alpha)}}(z) = \frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)} (1_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta} 1_{(1,\infty)}(\|z\|)) \leq 1.$$

Hence, the rejection method gives the H-sequence as above. Then,

$$\int_{i-1}^{i} E\left[\left(\frac{\alpha s}{\lambda_q} \right)^{-1/\alpha} 1(g((\alpha s/\lambda_q)^{-1/\alpha} V_1) \geq U_1) V_1 1((\alpha s/\lambda_q)^{-1/\alpha} \leq 1) \right] ds = c_i$$

gives the centering constants.

It seems that the centering constants have no closed form. But if $\nu_{TlS}^{\alpha,\beta}$ is symmetric, then $\int_{S^{d-1}} q(r, \xi)\xi\lambda(d\xi) = 0$ for every $r \in (0, \infty)$, and thus $c_i \equiv 0$. We also note that a different form can be derived by using $d\nu_{TlS}^{\alpha,\beta}/d\nu_{(\beta)}$.

4.2 Mixed Stable Processes

Definition 4.2.1. *A probability measure* μ *on* $\mathbb{R}^d$ *is called mixed stable if it is infinitely divisible with no Gaussian component and with Lévy measure* ν_{mS} *on* $\mathbb{R}_0^d$ *given* *by*

$$\nu_{mS}(B) = \int_{(0,2)} \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{\alpha+1}} \varphi(d\alpha), \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.2.1}$$

where λ *is a finite measure on* S^{d-1} *and* φ *is a probability measure on* $(0,2)$ *with no atoms satisfying*

$$\int_{(0,2)} \frac{1}{\alpha(2-\alpha)} \varphi(d\alpha) < \infty. \tag{4.2.2}$$

We have assumed that the measure φ has no atoms because each atom corresponds to an ordinary stable distribution and can be separated. Let us call φ the *stability measure*.

In the preceding section, we defined the classes $L_m(\mathbb{R}^d)$, for $m = 0, 1, \ldots$. Let us also define the class $L_\infty(\mathbb{R}^d)$ of their intersection, $L_\infty(\mathbb{R}^d) := \cap_{m=0}^\infty L_m$. Theorem 3.4 of Sato [44] shows that an infinitely divisible probability measure μ is in the class L_∞ if and only if its Lévy measure has the form (4.2.1). Moreover, recall that the class $L_0(\mathbb{R}^d)$ can be defined by its Lévy measure of the form (4.1.2). Barndorff-Nielsen et al.[4] recently define the subclass $T(\mathbb{R}^d)$ of $L_0(\mathbb{R}^d)$ by further requiring the completely monotone property of the function $k_\xi(r)$ in r for σ-a.e. ξ. Mixed stable distributions are also in the class T because $\int_{(0,2)} r^{-\alpha} \varphi(d\alpha)$ is completely monotone in r.

It is easy to derive the characteristic function of mixed stable distributions under some conditions on the stability measure.

Proposition 4.2.2. *If* $\varphi(0,1) = 0$ *or* $\varphi(1,2) = 0$*, then the characteristic function* $\widehat{\mu}$ *of a mixed stable distribution with Lévy measure (4.2.1) is given by*

$$\begin{aligned} \widehat{\mu}(y) \quad = \quad \exp\Bigg[& \int_{(0,2)} \Gamma(-\alpha) \cos\frac{\pi\alpha}{2} \int_{S^{d-1}} |\langle y, \xi \rangle|^\alpha \left(1 - i \tan\frac{\pi\alpha}{2} \operatorname{sgn}\langle y, \xi\rangle\right) \lambda(d\xi) \varphi(d\alpha) \\ & + i\langle y, b\rangle \Bigg], \end{aligned} \tag{4.2.3}$$

for some $b \in \mathbb{R}^d$.

A finite measure σ on S^{d-1} is said to be rotation invariant if $\sigma(B) = \sigma(U^{-1}B)$ for every orthogonal matrix U, where $U^{-1}B = \{U^{-1}x : x \in B\}$. If λ is rotation invariant and $b = 0$ in (4.2.3), then the characteristic function reduces to

$$\widehat{\mu}(y) = \exp\left[-C \int_{(0,2)} |y|^\alpha \widetilde{\varphi}(d\alpha)\right] = \exp[-\psi(y^2/2)],$$

where $\widetilde{\varphi}(d\alpha) = -\Gamma(-\alpha)\cos(\pi\alpha/2)\varphi(d\alpha)$, $C = \int_{S^{d-1}} |\langle \zeta, \xi \rangle|^\alpha \lambda(d\xi)$, $\zeta \in S^{d-1}$ (notice that C is constant in ζ because λ is rotation invariant), and

$$\psi(y) = C \int_{(0,2)} |2y|^{\alpha/2} \widetilde{\varphi}(d\alpha).$$

Since $\psi(0) = 0$ and ψ' is completely monotone on $(0, \infty)$, such distribution is of type G by Proposition 3 of Rosiński [39]. (Type G distributions can be thought of as marginals of a Brownian motion subordinated to an infinitely divisible process with positive jumps.)

The following is the moment properties of mixed stable distributions.

Proposition 4.2.3. *Let μ be a mixed stable distribution with Lévy measure (4.2.1). Then, for $p > 0$, $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ if and only if*

$$\int_1^\infty \frac{dr}{r^{\alpha+1-p}} \varphi(d\alpha) < \infty.$$

Proof. Since

$$\int_{\|z\|>1} \|z\|^p \nu_{mS}(dz) = \lambda(S^{d-1}) \int_{(0,2)} \int_1^\infty \frac{dr}{r^{\alpha+1-p}} \varphi(d\alpha),$$

we get the result by Theorem 1.2.2.

In the sequel, we say that $\{X_t^{mS} : t \geq 0\}$ is a mixed stable process if it is a Lévy process without Gaussian component and with the Lévy measure (4.2.1).

The next theorem gives a series representation of mixed stable processes. Unlike stable processes, the Inverse Lévy measure method is not applicable due to the extra complexity in structure caused by the stability measure φ. The following is derived by the generalized shot noise method.

Theorem 4.2.4. *Let* $\{X_t^{mS} : t \in [0,1]\}$ *be a mixed stable process in* $\mathbb{R}^d$ *whose characteristic function is given by*

$$E[e^{i\langle y, X_1^{mS}\rangle}] = \begin{cases} \exp\left[\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle}-1)\nu_{mS}(dz)\right], & if\ \varphi(1,2)=0, \\ \exp\left[\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle}-1-i\langle y,z\rangle)\nu_{mS}(dz)\right], & if\ \varphi(0,1)=0 \\ \exp\left[\int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle}-1-i\langle y,z\rangle 1_{\|z\|\le 1}(z))\nu_{mS}(dz)\right], & otherwise. \end{cases} \tag{4.2.4}$$

Then,

$$\{X_t^{mS} : t \in [0,1]\} \stackrel{d}{=} \left\{\sum_{i=1}^{\infty}[J(\Gamma_i,\alpha_i)V_i 1(T_i \le t) - c_i z_0 t] + b z_0 t : t \in [0,1]\right\}, \tag{4.2.5}$$

where the equality holds for finite dimensional distributions, the infinite sum on the right hand side converges a.s. uniformly on $[0,1]$. *Here,*

$$J(s,\alpha) = \left(\frac{\alpha s}{\lambda(S^{d-1})}\right)^{-1/\alpha},$$

and $\{\Gamma_i\}_{i\ge1}$ *are arrival times of a standard Poisson process,* $\{\alpha_i\}_{i\ge1}$ *is an sequence of iid random variables with common distribution* φ, $\{T_i\}_{i\ge1}$ *is iid uniform in* $[0,1]$, $\{V_i\}_{i\ge1}$ *is a sequence of iid random vectors in* S^{d-1} *with the common distribution* σ *defined by*

$$\sigma(C) = \lambda(C)/\lambda(S^{d-1}), \quad C \in \mathcal{B}(S^{d-1}),$$

and $z_0 = \int_{S^{d-1}}\xi\lambda(d\xi)/\lambda(S^{d-1})$. *Moreover, the random sequences* $\{T_i\}_{i\ge1}$, $\{\alpha_i\}_{i\ge1}$, $\{\Gamma_i\}_{i\ge1}$, *and* $\{V_i\}_{i\ge1}$ *are all mutually independent. Finally,* $\{c_i\}_{i\ge1}$ *is a sequence of constants given by*

$$c_i = \begin{cases} 0, & if\ \varphi(1,2)=0, \\ \int_{(1,2)}\left(\frac{\alpha i}{\lambda(S^{d-1})}\right)^{-1/\alpha}\varphi(d\alpha), & if\ \varphi(0,1)=0, \\ \int_{(0,2)}\left[\left(\frac{\alpha i}{\lambda(S^{d-1})}\right)^{-1/\alpha} - \frac{\lambda(S^{d-1})}{\alpha-1}\right]\varphi(d\alpha), & otherwise, \end{cases}$$

and

$$b = \begin{cases} 0, & if \ \varphi(1,2) = 0, \\ \int_{(0,2)} \left(\frac{\alpha}{\lambda(S^{d-1})}\right)^{-1/\alpha} \zeta(-1/\alpha)\varphi(d\alpha), & otherwise. \end{cases}$$

Proof. The H-sequence can be obtained by the generalized shot noise method based on the disintegration of ν_{mS};

$$\begin{aligned} \nu_{mS}((b,\infty)C) &= \int_{(0,2)} \int_{S^{d-1}} \lambda(d\xi) \int_0^\infty 1_{(b,\infty)C}(r\xi) \frac{dr}{r^{\alpha+1}} \varphi(d\alpha) \\ &= \int_{(0,2)} \int_{S^{d-1}} \int_0^\infty 1_{(b,\infty)C}((\alpha r/\lambda(S^{d-1}))^{-1/\alpha}\xi) dr\, \sigma(d\xi)\varphi(d\alpha). \end{aligned}$$

In view of (4.2.4), the case $\varphi(1,2) = 0$ is proved. For the case $\varphi(1,2) \neq 0$, letting

$$d_i = \begin{cases} \int_{i-1}^i E\left[\left(\frac{\alpha_1 s}{\lambda(S^{d-1})}\right)^{-1/\alpha_1}\right] ds, & if \ \varphi(0,1) = 0, \\ \int_{i-1}^i E\left[\left(\frac{\alpha_1 s}{\lambda(S^{d-1})}\right)^{-1/\alpha_1} 1\left(\left(\frac{\alpha_1 s}{\lambda(S^{d-1})}\right)^{-1/\alpha_1} \leq 1\right)\right] ds, & otherwise, \end{cases}$$

we show that $\sum_{i=1}^\infty (c_i - d_i) = b$. If $\varphi(0,1) = 0$,

$$\begin{aligned} \sum_{i=1}^n (c_i - d_i) &= \sum_{i=1}^n \int_{(1,2)} \left(\frac{\alpha\, i}{\lambda(S^{d-1})}\right)^{-1/\alpha} \varphi(d\alpha) - \int_0^n \int_{(1,2)} \left(\frac{\alpha\, s}{\lambda(S^{d-1})}\right)^{-1/\alpha} \varphi(d\alpha) ds \\ &= \int_{(1,2)} \left(\frac{\alpha}{\lambda(S^{d-1})}\right)^{-1/\alpha} \left[\sum_{i=1}^n i^{-1/\alpha} - \int_0^n s^{-1/\alpha} ds\right] \varphi(d\alpha) \to b, \end{aligned}$$

as $n \to \infty$, where the passage to the limit is justified by the monotone convergence theorem. Finally, if $\varphi(0,1) \neq 0$, observing that

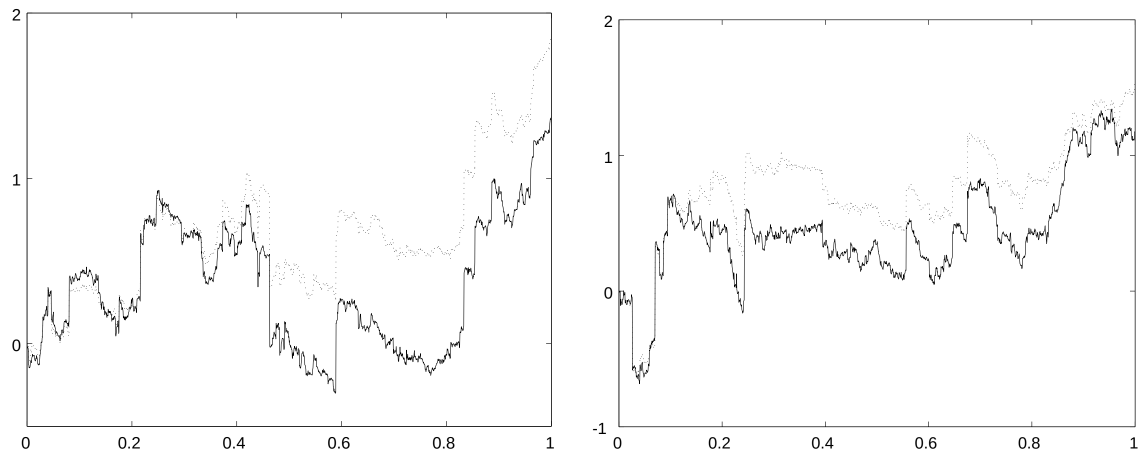
$$\begin{aligned} &\sum_{i=1}^n \int_{i-1}^i E\left[\left(\frac{\alpha_1 s}{\lambda(S^{d-1})}\right)^{-1/\alpha_1} 1\left(\left(\frac{\alpha_1 s}{\lambda(S^{d-1})}\right)^{-1/\alpha_1} \leq 1\right)\right] ds \\ &\quad = E\left[\left(\frac{\alpha_1}{\lambda(S^{d-1})}\right)^{-1/\alpha_1} \frac{\alpha_1}{\alpha_1 - 1}\left(\left(n \vee \frac{\lambda(S^{d-1})}{\alpha_1}\right)^{1-1/\alpha_1} - \left(\frac{\lambda(S^{d-1})}{\alpha_1}\right)^{1-1/\alpha_1}\right)\right], \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i=1}^n (c_i - d_i) \\ &= \int_{(0,2)} \left(\frac{\alpha}{\lambda(S^{d-1})}\right)^{-1/\alpha} \left[\sum_{i=1}^n i^{-1/\alpha} - \frac{1}{1 - 1/\alpha_1}\left(n \vee \frac{\lambda(S^{d-1})}{\alpha_1}\right)^{1-1/\alpha_1}\right] \varphi(d\alpha) \to b, \end{aligned}$$

as $n \to \infty$, where the interchange of the limit with the integral holds by the monotone convergence theorem. The proof is complete.

The structure of the series representation tells us that mixed stable processes can be thought of as a mixture of stable processes of random stability indices with distribution φ. In Figure 4.2, we give typical sample paths of mixed stable processes, generated by the series representation.



$\{\alpha_j\} \sim U(1.0, 1.8)$ $\{\alpha_j\} \sim U\{1.0, 1.2, 1.6, 1.8\}$

Figure 4.2: Typical sample paths of mixed stable process (—) and 1.4-stable process (- -)

Definition 4.2.5. *A probability measure μ on $\mathbb{R}^d$ is called tempered mixed stable if it is infinitely divisible with no Gaussian component and with Lévy measure ν_{TmS} on $\mathbb{R}^d_0$ given by*

$$\nu_{TmS}(B) = \int_{(0,2)} \int_{\mathbb{R}^d_0} \int_0^\infty 1_B(sx) s^{-\alpha-1} e^{-s} ds\, \rho(dx)\, \varphi(d\alpha), \quad B \in \mathcal{B}(\mathbb{R}^d_0), \tag{4.2.6}$$

where ρ is a σ-finite Borel measure on $\mathbb{R}^d_0$ and φ is a probability measure on $(0,2)$ with no atoms satisfying

$$\int_{(0,2)} \int_{\mathbb{R}^d_0} \frac{\|x\|^\alpha}{\alpha(2-\alpha)} \rho(dx) \varphi(d\alpha) < \infty. \tag{4.2.7}$$

Notice that a Lévy measure without the exponential tempering term e^{-s} in (4.2.6) is *not* equivalent to Lévy measure ν_{mS} of mixed stable distribution given in (4.2.1). It is worth mentioning that (4.2.7) is a sufficient condition for ν_{TmS} to be well defined as a Lévy measure. It is indeed the necessary and sufficient condition for $\bar{\nu}_{mS}$ to be a Lévy measure where

$$\bar{\nu}_{mS}(B) = \int_{(0,2)} \int_{\mathbb{R}_0^d} \int_0^\infty 1_B(sx) s^{-\alpha-1} ds\, \rho(dx)\, \varphi(d\alpha), \quad B \in \mathcal{B}(\mathbb{R}_0^d). \tag{4.2.8}$$

Clearly, the class of tempered mixed stable distributions contains tempered stable distributions. Moreover, as in the case of tempered stable distributions, the Lévy measure of tempered mixed stable distributions admits an equivalent polar coordinate form.

Proposition 4.2.6. *ν_{TmS} is a Lévy measure of a tempered mixed stable distribution if and only if in polar coordinates it has the form*

$$\nu_{TmS}(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) \frac{q(r,\xi)}{r} \left(\int_{(0,2)} r^{-\alpha} \varphi(d\alpha) \right) dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.2.9}$$

where $q : (0,\infty) \times S^{d-1} \to (0,\infty)$ is a Borel function such that $q(\cdot,\xi)$ is completely monotone with $\lim_{r\to\infty} q(r,\xi) = 0$ for every $\xi \in S^{d-1}$, σ is a probability measure on S^{d-1} such that

$$\int_{S^{d-1}} q(0+,\xi)\sigma(d\xi) < \infty,$$

and φ is a probability measure on $(0,2)$ such that

$$\int_{(0,2)} \frac{1}{\alpha(2-\alpha)} \varphi(d\alpha) < \infty.$$

The proof is similar to that of Proposition 2.2.3. By the same argument as in the mixed stable case, a tempered mixed stable distribution is selfdecomposable, and moreover, it is in class T.

Proposition 4.2.7. *Let μ be a tempered mixed stable distribution with Lévy measure (4.2.6). Set $\underline{\alpha} := \inf\{\alpha \in (0,2) : \varphi(\alpha,2) > 0\}$ and $\overline{\alpha} := \sup\{\alpha \in (0,2) : \varphi(\alpha,2) > 0\}$. Then,*

(i) For $p \in (0,\underline{\alpha})$, $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$.

(ii) For $p > \overline{\alpha}$, $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$ if and only if $\int_{\|x\|>1} \|x\|^p \rho(dx)$.

(iii) For $p \in [\underline{\alpha},\overline{\alpha}]$, if $\int_{\|x\|>1} \|x\|^p \ln \|x\| \rho(dx) < \infty$, $\int_{(p,2)} \int_{\mathbb{R}^d_0} \frac{\|x\|^\alpha}{\alpha - p} \rho(dx)\varphi(d\alpha) < \infty$ and $\int_{(0,p)} \Gamma(p-\alpha)\varphi(d\alpha) < \infty$, then $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$.

(iii) if $\rho(\{x : \|x\| > \epsilon\}) = 0$ for some $\epsilon > 0$, then for every $\theta \in (0,\epsilon^{-1})$,

$$\int_{\mathbb{R}^d} \exp(\theta\|x\|)\mu(dx) < \infty.$$

Proof. Let us first note that (4.2.7) implies that $\int_{(0,2)} \int_{\mathbb{R}^d_0} \|x\|^\alpha \rho(dx)\varphi(d\alpha) < \infty$. For notational convenience, we will use the exponential integral function

$$E_r(x) = x^{r-1} \int_x^\infty s^{-r} e^{-s} ds, \quad r \in \mathbb{R},\ x \in (0,\infty).$$

The claim (i) follows easily from

$$\begin{aligned} \int_{\|z\|\geq 1} \|z\|^p \nu_{TmS}(dz) &= \int_{(0,2)} \int_{\mathbb{R}^d_0} \|x\|^p \int_{1/\|x\|} s^{p-\alpha-1} e^{-s} ds \rho(dx)\varphi(d\alpha) \\ &\leq \frac{1}{\underline{\alpha} - p} \int_{(0,2)} \int_{\mathbb{R}^d_0} \|x\|^\alpha \rho(dx)\varphi(d\alpha) < \infty. \end{aligned}$$

For (ii) and (iii), the proof will be based on the decomposition

$$\begin{aligned} \int_{\|z\|\geq 1} \|z\|^p \nu_{TmS}(dz) &= \int_{(0,2)} \int_{\|x\|\leq 1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|) \rho(dx)\varphi(d\alpha) \\ &\quad + \int_{(0,2)} \int_{\|x\|>1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|) \rho(dx)\varphi(d\alpha). \end{aligned} \tag{4.2.10}$$

Let us consider (ii). The first term of the left hand side of (4.2.10) is bounded because on $\{\|x\| \leq 1\}$, $E_{-p+\alpha+1}(1/\|x\|) \leq E_{-p+\alpha+1}(1) < \infty$. Hence, the second term determines the finiteness of the p-th moment. The second term can be written as

$$\int_{(0,2)} \int_{\|x\|>1} \|x\|^p \int_{1/\|x\|} s^{p-\alpha-1} e^{-s} ds \rho(dx)\varphi(d\alpha),$$

and observe that for $\|x\| > 1$,

$$\int_1^\infty s^{p-\alpha-1}e^{-s}ds \le \int_{1/\|x\|} s^{p-\alpha-1}e^{-s}ds \le \Gamma(p-\alpha).$$

Therefore, the second term is finite if and only if $\int_{\|x\|>1} \|x\|^p \rho(dx) < \infty$, which proves (ii). To prove (iii), we decompose the first term of (4.2.10) into two parts;

$$\int_{(0,p)} \int_{\|x\|\le 1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|)\rho(dx)\varphi(d\alpha)$$
$$+ \int_{(p,2)} \int_{\|x\|\le 1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|)\rho(dx)\varphi(d\alpha),$$

where the first is bounded by $E_1(1) \int_{(0,p)} \int_{\|x\|\le 1} \|x\|^\alpha \rho(dx)\varphi(d\alpha) < \infty$ and the second by $\int_{(p,2)} \int_{\|x\|\le 1} \|x\|^\alpha/(\alpha - p)\rho(dx)\varphi(d\alpha) < \infty$. Let us now consider the second term of (4.2.10). On $\{\|x\| > 1\}$, we have that $0.5 \ln \|x\| \le \int_{1/\|x\|}^\infty s^{-1}e^{-s}ds \le \int_1^\infty s^{-1}e^{-s}ds + \ln \|x\|$. This implies that $\int_{\|x\|>1} \|x\|^p \ln \|x\| \rho(dx) < \infty$ is equivalent to $\int_{\|x\|>1} \|x\|^p E_1(1/\|x\|)\rho(dx) < \infty$. This and the fact that φ has no atoms justify the following decomposition of the second term of (4.2.10);

$$\int_{(0,p)} \int_{\|x\|>1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|)\rho(dx)\varphi(d\alpha)$$
$$+ \int_{(p,2)} \int_{\|x\|>1} \|x\|^\alpha E_{-p+\alpha+1}(1/\|x\|)\rho(dx)\varphi(d\alpha),$$

where the second is bounded by $\int_{(p,2)} \int_{\|x\|>1} \|x\|^\alpha/(\alpha-p)\rho(dx)\varphi(d\alpha) < \infty$ and the first is bounded by $\int_{\|x\|>1} \|x\|^p \rho(dx) \int_{(0,p)} \Gamma(p-\alpha)\varphi(d\alpha)$. The finiteness of $\int_{\|x\|>1} \|x\|^p \rho(dx)$ is guaranteed by $\int_{\|x\|>1} \|x\|^p \ln \|x\| \rho(dx) < \infty$. Thus, the bound is also finite. Finally, the proof for (iv) is similar to that in Proposition 2.2.8.

We call a non-Gaussian Lévy process in $\mathbb{R}^d$ with Lévy measure ν_{TmS} on $\mathbb{R}^d_0$ a tempered mixed stable process.

Proposition 4.2.8. *Let μ be a tempered mixed stable distribution on $\mathbb{R}^d$. Assume either $\varphi((0,1]) = 0$ or $\varphi([1,2)) = 0$. Then,*

$$\widehat{\mu}(y) = \exp\left[\int_{(0,2)} \Gamma(-\alpha) \int_{\mathbb{R}^d_0} \psi(\langle y, x\rangle, \alpha)\rho(dx)\, \varphi(d\alpha) + i\langle y, \gamma\rangle\right], \tag{4.2.11}$$

where $\gamma \in \mathbb{R}^d$ *and* $\psi : \mathbb{R} \times (0,2) \to C$ *is given by*

$$\psi(s,\alpha) = \begin{cases} (1-is)^\alpha - 1, & if\ \varphi([1,2)) = 0, \\ (1-is)^\alpha - 1 + i\alpha s, & if\ \varphi((0,1]) = 0. \end{cases} \tag{4.2.12}$$

We will call a Lévy process in $\mathbb{R}^d$ generated by $(\gamma, 0, \nu_{TmS})$ a tempered mixed stable process. For convenience, let us denote by $\{X_t^{TmS} : t \geq 0\}$ a tempered mixed stable process whose characteristic function at time 1 is given by (4.2.11) with $\gamma = 0$ when either $\varphi(0,1) = 0$ or $\varphi(1,2) = 0$; when no condition is imposed on φ, its characteristic function is to be given by

$$\widehat{\mu}(y) = \exp\left[\int_{\mathbb{R}_0^d} (e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle 1_{\|z\|\leq 1}(z))\nu_{TmS}(dz)\right].$$

In this setting, we will prove a long-time asymptotic Gaussianity of tempered mixed stable processes below.

Theorem 4.2.9. *Assume* $\int_{\mathbb{R}_0^d} \|x\|^2 \rho(dx) < \infty$*. When* $\varphi(1,2) = 0$*, assume further that* $\int_{(0,1)} \Gamma(1-\alpha)\varphi(d\alpha) < \infty$ *and* $\int_{\mathbb{R}_0^d} \|x\|\rho(dx) < \infty$*. Otherwise, assume that* $\int_{(0,1)} \Gamma(1-\alpha)\varphi(d\alpha) < \infty$, $\int_{(1,2)} \int_{\mathbb{R}_0^d} \|x\|^\alpha/(\alpha-1)\rho(dx)\varphi(d\alpha) < \infty$ *and* $\int_{\|x\|>1} \|x\| \ln \|x\| \rho(dx) < \infty$*. Then,*

$$\{h^{-1/2}(X_{ht}^{TmS} - b\,ht) : t \geq 0\} \xrightarrow{d} \{B_t : t \geq 0\} \quad as \quad h \to \infty, \tag{4.2.13}$$

where $\{B_t : t \geq 0\}$ *is a Brownian motion with characteristic function*

$$E[e^{i\langle y,B_t\rangle}] = \exp\left[-\frac{t}{2}\int_{(0,2)} \Gamma(2-\alpha)\varphi(d\alpha)\int_{\mathbb{R}_0^d} \langle y,x\rangle^2 \rho(dx)\right],$$

and

$$b = \begin{cases} \int_{(0,1)} \Gamma(1-\alpha)\varphi(d\alpha) \int_{\mathbb{R}_0^d} x\rho(dx), & if\ \varphi(1,2) = 0, \\ 0, & if\ \varphi(0,1) = 0, \\ -\int_{(0,2)} \int_{\mathbb{R}_0^d} \frac{x}{\|x\|^{\alpha-1}} E_\alpha(1/\|x\|)\rho(dx)\varphi(d\alpha), & otherwise. \end{cases}$$

Proof. It is enough to consider the marginal at $t = 1$. In view of Proposition 4.2.7, b is well defined and the random variable $h^{-1/2}(X_h^{TmS} - b\,h)$ is infinitely divisible generated by $(c, 0, h(T_{h^{-1/2}}\nu_{TmS}))$, where

$$c = -\int_{\|z\|>1} zh(T_{h^{-1/2}}\nu_{TmS})(dz).$$

As before, for $f : \mathbb{R}_0^d \to \mathbb{R}$ a bounded continuous function vanishing on a neighborhood of the origin, there is $\epsilon \in (0,1)$ such that $f(x) \equiv 0$ on $\{\|x\| \leq \epsilon\}$, and in view of Proposition 4.2.6, we have

$$\begin{aligned}
&\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz) \\
&\quad= \int_{S^{d-1}}\int_{(0,2)}\int_{\epsilon}^{\infty} f(r\xi)\frac{h^{(2-\alpha)/2}q(h^{1/2}r,\xi)}{r^{\alpha+1}}dr\varphi(d\alpha)\sigma(d\xi) \\
&\quad= \int_{S^{d-1}}\int_{(0,2)}\int_{\epsilon}^{\infty} \frac{f(r\xi)}{r^{\alpha+1}}\int_{(0,\infty)} h^{(2-\alpha)/2}e^{-h^{1/2}rx}\rho_\xi(dx)dr\varphi(d\alpha)\sigma(d\xi) \to 0,
\end{aligned}$$

as $h \to \infty$, where the interchange of the limit with the integral is justified by

$$\begin{aligned}
&\left|\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{lS}^{\alpha,\beta})(dz)\right| \\
&\quad\leq \int_{S^{d-1}}\int_{(0,2)}\int_{\epsilon}^{\infty} \frac{|f(r\xi)|}{r^{\alpha+1}}\int_{(0,\infty)} h^{(2-\alpha)/2}e^{-h^{1/2}rx}\rho_\xi(dx)dr\varphi(d\alpha)\sigma(d\xi) \\
&\quad\leq C\int_{S^{d-1}}\int_{(0,2)}\frac{\epsilon^{-\alpha}}{\alpha}\int_{(0,\infty)} e^{-h^{1/2}\epsilon x}\rho_\xi(dx)\varphi(d\alpha)\sigma(d\xi) \\
&\quad= C\int_{S^{d-1}} q(\epsilon,\xi)\sigma(d\xi)\int_{(0,2)}\frac{\epsilon^{-\alpha}}{\alpha}\varphi(d\alpha) \\
&\quad\leq \frac{C}{\epsilon^2}\int_{S^{d-1}} q(0+,\xi)\sigma(d\xi)\int_{(0,2)}\frac{1}{\alpha(2-\alpha)}\varphi(d\alpha) < \infty.
\end{aligned}$$

Finally, for every $\kappa > 0$,

$$\int_{\|z\|\leq\kappa} zz^T h(T_{h^{-1/2}}\nu_{TmS})(dz) = \int_{\|z\|\leq h^{1/2}\kappa} zz^T\nu_{TmS}(dz) \to \int_{\mathbb{R}_0^d} zz^T\nu_{TmS}(dz),$$

and $\int_{\|z\|>\kappa} zh(T_{h^{-1/2}}\nu_{TmS})(dz) \to 0$, as $h \to \infty$, which concludes the proof.

The structure of ν_{TmS} tells us that its disintegration is a simple generalization of that of tempered stable Lévy measure. In view of Proposition 2.4.1 and 2.4.2, the

H-sequences will be easily obtained by adding randomness on the stability index α with distribution φ. Below, we will state two forms of series representations of a tempered mixed stable processes; one by the rejection method and the other by the generalized shot noise method. It seems that centering constants have no nice closed form. We will also give those for the sake of completeness.

Theorem 4.2.10. *Let $\{X_t^{TmS} : t \in [0,1]\}$ be a tempered mixed stable process in $\mathbb{R}^d$ whose characteristic function is given by (4.2.11). Then,*

$$\{X_t^{TmS} : t \in [0,1]\} \stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[J_i^0 1(g(J_i^0) \le Y_i) 1(T_i \le t) - c_i t \right] + at : t \in [0,1] \right\}, \tag{4.2.14}$$

or,

$$\{X_t^{TmS} : t \in [0,1]\} \stackrel{d}{=} \left\{ \sum_{i=1}^{\infty} \left[J_i^1 1(T_i \le t) - d_i \right] + bt : t \in [0,1] \right\}, \tag{4.2.15}$$

where the equalities hold for finite dimensional distributions. Here, $\{T_i\}_{i\ge1}$ is given in Theorem 4.2.4, $\{Y_i\}_{i\ge1}$ is iid uniform on $[0,1]$, and $g : \mathbb{R}_0^d \to [0,1)$ is given by

$$g(z) = \begin{cases} \frac{q(\|z\|, z/\|z\|)}{q(0+, z/\|z\|)}, & if \;\; q(0+, z/\|z\|) > 0, \\ 1, & otherwise, \end{cases}$$

where $q : (0,\infty) \times S^{d-1} \to (0,\infty)$ is given as in Proposition 4.2.6. Moreover, $\{J_i^0\}_{i\ge1}$ and $\{J_i^1\}_{i\ge1}$ are given by

$$\begin{aligned} J_i^0 &= m(\alpha_i, \rho)(\alpha_i \Gamma_i)^{-1/\alpha_i} \frac{V_i}{\|V_i\|}, \\ J_i^1 &= \left[m(\alpha_i, \rho)(\alpha_i \Gamma_i)^{-1/\alpha_i} \wedge E_i U_i^{1/\alpha_i} \|V_i\| \right] \frac{V_i}{\|V_i\|}, \end{aligned}$$

where $\{\Gamma_i\}_{i\ge1}$, $\{\alpha_i\}_{i\ge1}$ are defined as in Theorem 4.2.4, $\{E_i\}_{i\ge1}$ is iid exponential with parameter 1, $\{U_i\}_{i\ge1}$ is iid uniform in $[0,1]$, and $m(\alpha,\rho) = (\int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx))^{1/\alpha}$. Given $\{\alpha_i\}_{i\ge1}$, $\{V_i\}_{i\ge1}$ is a sequence of independent random vectors in $\mathbb{R}_0^d$ with distribution ρ_{α_i} defined by

$$\rho_\alpha(dx) = \frac{\|x\|^\alpha}{m(\alpha,\rho)^\alpha} \rho(dx).$$

Finally, $\{c_i\}_{i\geq 1}$ *and* $\{d_i\}_{i\geq 1}$ *are sequences of constants given by*

$$c_i = \begin{cases} 0, & if\ \varphi(1,2)=0, \\ \int_{(0,2)} \left[m(\alpha,\rho)(\alpha i)^{-1/\alpha} - m(\alpha,\rho)^2(\alpha i)^{-2/\alpha}\right] & \\ \qquad \frac{\int_{\mathbb{R}_0^d} x\|x\|^{\alpha-1}\rho(dx)}{m(\alpha,\rho)^\alpha}\varphi(d\alpha), & if\ \varphi(0,1)=0, \\ \int_{(0,2)} \int_{(i-1)\vee\frac{m(\alpha,\rho)^\alpha}{\alpha}}^{i\vee\frac{m(\alpha,\rho)^\alpha}{\alpha}} m(\alpha,\rho)(\alpha s)^{-1/\alpha}(1-g(m(\alpha,\rho)(\alpha s)^{-1/\alpha}))ds & \\ \qquad \frac{\int_{\mathbb{R}_0^d} x\|x\|^{\alpha-1}\rho(dx)}{m(\alpha,\rho)^\alpha}\varphi(d\alpha), & otherwise, \end{cases}$$

$$d_i = \begin{cases} 0, & if\ \varphi(1,2)=0, \\ \int_{(0,2)} m(\alpha,\rho)^{1-\alpha}(\alpha i)^{-1/\alpha}\int_{\mathbb{R}_0^d} x\|x\|^{\alpha-1}\rho(dx)\varphi(d\alpha), & if\ \varphi(0,1)=0, \\ \int_{i-1}^{i} E\Big[(m(\alpha_1,\rho)(\alpha_1 s)^{-1/\alpha}\wedge E_1U_1^{1/\alpha_1}\|V_1\|) & \\ \qquad 1(m(\alpha_1,\rho)(\alpha_1 s)^{-1/\alpha}\wedge E_1U_1^{1/\alpha_1}\|V_1\|)\leq 1)\frac{V_1}{\|V_1\|}\Big]ds, & otherwise, \end{cases}$$

and

$$a = \begin{cases} \int_{(0,2)} m(\alpha,\rho)\alpha^{-1/\alpha}(\zeta(-1/\alpha) - m(\alpha,\rho)\alpha^{-1/\alpha}\zeta(-2/\alpha))\varphi(d\alpha), & if\ \varphi(0,1)=0, \\ 0, & otherwise, \end{cases}$$

$$b = \begin{cases} \int_{(0,2)} \Big[m(\alpha,\rho)^{1-\alpha}\alpha^{-1/\alpha}\zeta(-1/\alpha)\int_{\mathbb{R}_0^d} x\|x\|^{\alpha-1}\rho(dx) & \\ \qquad +|\Gamma(1-\alpha)|\int_{\mathbb{R}_0^d}\rho(dx)\Big]\varphi(d\alpha), & if\ \varphi(0,1)=0, \\ 0, & otherwise. \end{cases}$$

4.3 Concluding Remarks

We have defined layered and mixed stable processes and studied their properties. Layered stable processes possess short-time and long-time behaviors as tempered stable processes. It is worth noting that the appearance of the sample paths of layered stable processes are, in general, different in a regular time or a short time while tempered α-stable processes originally look like their short-time α-stable processes, as seen in Figure 2.1 and Figure 2.3.

Chapter 5

Empirical Studies

5.1 A Variance Reduction Method in Monte Carlo Simulation via Esscher Transform

In this section, we present a method to achieve a faster convergence in the Monte Carlo simulation with the help of a special class of the density transformation of Lévy processes, so called the *Esscher transform*. Let us first state it in the multi-dimensional setting.

Lemma 5.1.1. *Let $\{X_t : t \geq 0\}$ be a Lévy process in $\mathbb{R}^d$ generated by $(\gamma, 0, \nu)_1$ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathcal{F}_t := \sigma(X_s; s \in [0,t])$. Assume that there exists $\lambda \in \mathbb{R}^d$ such that $E[e^{\langle \lambda, X_1\rangle}] < \infty$. Then, on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ where Q is defined by*

$$dQ|_{\mathcal{F}_t} = \frac{e^{\langle \lambda, X_t\rangle}}{E[e^{\langle \lambda, X_t\rangle}]} dP|_{\mathcal{F}_t}, \tag{5.1.1}$$

$\{X_t : t \geq 0\}$ is a Lévy process generated by $(\gamma_\lambda, 0, \nu_\lambda)_1$ where

$$\gamma_\lambda = \gamma + \int_{\mathbb{R}_0^d} z(e^{\langle \lambda, z\rangle} - 1)\nu(dz) \quad \text{and} \quad \nu_\lambda(dz) = e^{\langle \lambda, z\rangle}\nu(dz). \tag{5.1.2}$$

The Esscher transform has been widely used in mathematical finance to derive an equivalent probability measure on which a discounted asset price process is a martingale so as to evaluate a fair option price with no chance of arbitrage. Consider a Lévy process $\{X_t : t \geq 0\}$ in $\mathbb{R}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$. Define a probability measure

P_λ, equivalent to Q, by

$$\frac{dP_\lambda}{dQ}|_{\mathcal{F}_t} := \frac{e^{\lambda X_t}}{E_Q[e^{\lambda X_t}]}, \tag{5.1.3}$$

where $\lambda \in \mathbb{R}$ is such that $E_Q[e^{\lambda X_1}] < \infty$, and set $\Lambda := \{\lambda \in \mathbb{R} : E_Q[e^{\lambda X_1}] < \infty\}$. Clearly, the stochastic process $\{dP_\lambda/dQ|_{\mathcal{F}_t} : t \geq 0\}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ under the probability measure Q. It is also known that the equivalence of P_λ and Q implies

$$\frac{dQ}{dP_\lambda}|_{\mathcal{F}_t} = \left(\frac{e^{\lambda X_t}}{E_Q[e^{\lambda X_t}]}\right)^{-1}.$$

The following lemma is useful.

Lemma 5.1.2. *Let p and p_λ be a density function of X_T, respectively, under Q and P_λ defined by (5.1.3) for some $\lambda \in \Lambda$. Then,*

$$p_\lambda(x) = \frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]} p(x), \quad x \in \mathbb{R}. \tag{5.1.4}$$

Proof. It is enough to justify the result for f continuous with compact support. We have

$$\begin{aligned} E_Q[f(X_T)] &= E_{P_\lambda}\left[\frac{dQ}{dP_\lambda}|_{\mathcal{F}_T} f(X_T)\right] \\ &= E_{P_\lambda}\left[\left(\frac{e^{\lambda X_T}}{E_Q[e^{\lambda X_T}]}\right)^{-1} f(X_T)\right] \\ &= \int_{\mathbb{R}} f(x) \left(\frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]}\right)^{-1} p_\lambda(x)dx. \end{aligned} \tag{5.1.5}$$

Since $E_Q[f(X_T)] = \int_{\mathbb{R}} f(x)p(x)dx$, we get (5.1.4). Moreover, $(e^{\lambda x}/E_Q[e^{\lambda X_T}])p(x)$ is well defined as a density because $e^{\lambda x}/E_P[e^{\lambda X_T}] > 0$ and

$$\int_{\mathbb{R}} \frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]} p(x)dx = \frac{E_Q[e^{\lambda X_T}]}{E_Q[e^{\lambda X_T}]} = 1,$$

which concludes the proof.

From its construction (5.1.4), the density p_λ tilts p to the right when $\lambda > 0$. Similarly, when $\lambda < 0$, it tilts p to the left. ($E_Q[e^{\lambda X_T}]$ is essentially just a normalizing constant.) Roughly speaking, if $\lambda > 0$, a random variable X_T under P_λ tends to give greater values than under Q, while smaller if $\lambda < 0$.

Consider a stock price process $S_t := S_0 \exp(X_t)$, $t \geq 0$ under a real probability measure Q, where X is a Lévy process. Suppose that the risk-free rate is $r > 0$ and that under a probability measure P_λ defined by (5.1.3), the stock price process is a martingale. Such a measure P_λ is called an *equivalent martingale measure*, or a *risk neutral measure*. Then, the parameter $\lambda \in \Lambda$ must satisfy, for every $t \geq 0$, $E[e^{(\lambda+1)X_t}] < \infty$, and

$$e^{rt} = \frac{E_Q[e^{(\lambda+1)X_t}]}{E_Q[e^{\lambda X_t}]}, \tag{5.1.6}$$

or equivalently, $e^r = E_Q[e^{(\lambda+1)X_1}]/E_Q[e^{\lambda X_1}]$ by the independence and stationarity of increments. In fact,

$$E_{P_\lambda}[e^{-rt}S_t] = e^{-rt}E_Q\left[e^{X_t}\frac{e^{\lambda X_t}}{E_Q[e^{\lambda X_t}]}\right] = e^{-rt}\frac{E_Q[e^{(\lambda+1)X_t}]}{E_Q[e^{\lambda X_t}]} = 1.$$

Then, the European call option price $E_{P_\lambda}[e^{-rT}(S_T - K)_+]$ can be computed as

$$E_{P_\lambda}[e^{-rT}(S_T - K)_+] = S_0 \int_{\ln(K/S_0)}^{\infty} p_{\lambda+1}(x)dx - e^{-rT}K\int_{\ln(K/S_0)}^{\infty} p_\lambda(x)dx.$$

In fact, by (5.1.6) and Lemma 5.1.2,

$$\begin{aligned}
E_{P_\lambda}[e^{-rT}(S_T - K)^+] &= \int_{\ln(K/S_0)}^{\infty} e^{-rT}(S_0e^x - K)p_\lambda(x)dx \\
&= S_0\int_{\ln(K/S_0)}^{\infty} \frac{E_Q[e^{\lambda X_T}]}{E_Q[e^{(\lambda+1)X_T}]}e^x\frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]}p(x)dx \\
&\qquad\qquad -e^{-rT}K\int_{\ln(K/S_0)}^{\infty} p_\lambda(x)dx \\
&= S_0\int_{\ln(K/S_0)}^{\infty} \frac{e^{(\lambda+1)x}}{E_Q[e^{(\lambda+1)X_T}]}p(x)dx - e^{-rT}K\int_{\ln(K/S_0)}^{\infty} p_\lambda(x)dx \\
&= S_0\int_{\ln(K/S_0)}^{\infty} p_{\lambda+1}(x)dx - e^{-rT}K\int_{\ln(K/S_0)}^{\infty} p_\lambda(x)dx.
\end{aligned}$$

Let us now change our standpoint. Suppose that the stock price process $\{S_t : t \geq 0\}$ is a martingale under the probability measure Q and that the density p of X_T is known. We now want to evaluate $I := E_Q[f(X_T)]$ by Monte Carlo simulation. One usually generates iid random variables $\{x_i\}_{i\geq 1}$ with common density p, and compute $n^{-1}\sum_{i=1}^n f(x_i)$. By the strong law of large numbers, we have $n^{-1}\sum_{i=1}^n f(x_i) \to I$ a.s. as $n \to \infty$. The speed of convergence is usually measured in terms of the variance of the estimator, i.e.,

$$\mathrm{Var}_Q\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\right) = \frac{1}{n}\mathrm{Var}_Q(f(x_1)).$$

Clearly, the convergence tends to be faster with smaller $\mathrm{Var}_Q(f(x_1))$. We claim that this can be achieved by using the Esscher transform. Here, as a simple example, we take a path-independent case $f(X_T)$ for some f satisfying $E_Q[f(X_T)] < \infty$. The density p_λ of X_T under P_λ can be obtained by (5.1.4). In view of (5.1.5), we compute

$$\frac{1}{n}\sum_{i=1}^n f(x_i)\left(\frac{e^{\lambda x_i}}{E_Q[e^{\lambda X_T}]}\right)^{-1},$$

where $\{x_i\}_{i\geq 1}$ are iid random variables with common density p_λ. (Recall that p_λ is the density under P_λ, not p under Q.) Its variance is given by

$$\begin{aligned}
n\mathrm{Var}_{P_\lambda}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\left(\frac{e^{\lambda x_i}}{E_Q[e^{\lambda X_T}]}\right)^{-1}\right) &= \mathrm{Var}_{P_\lambda}\left(f(x_1)\left(\frac{e^{\lambda x_1}}{E_Q[e^{\lambda X_T}]}\right)^{-1}\right)\\
&= \int_{\mathbb{R}} f(x)^2\left(\frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]}\right)^{-2} p_\lambda(x)dx - I^2\\
&= \int_{\mathbb{R}} f(x)^2\left(\frac{e^{\lambda x}}{E_Q[e^{\lambda X_T}]}\right)^{-1} p(x)dx - I^2\\
&=: J(\lambda) - I^2.
\end{aligned}$$

We want to find λ minimizing the variance Var_{P_λ}, which is equivalent to the optimization problem $\mathrm{argmin}_{\lambda\in\Lambda} J(\lambda)$. Then, the following shows the convexity of J.

Lemma 5.1.3. *Suppose for $n = 1, 2$ and for $\lambda \in \Lambda$,*

$$\int_{\mathbb{R}^2} |x-y|^n e^{-\lambda(x-y)} p(x)p(y)f^2(x)dxdy < \infty. \tag{5.1.7}$$

Then, J is convex.

Proof. J can be written as

$$J(\lambda) = \int_{\mathbb{R}} e^{\lambda x} p(x)dx \int_{\mathbb{R}} f(x)^2 e^{-\lambda x} p(x)dx = \int_{\mathbb{R}^2} e^{-\lambda(x-y)} p(x)p(y)f(x)^2 dxdy.$$

By virtue of (5.1.7), we get

$$\frac{d^2}{d\lambda^2} J(\lambda) = \int_{\mathbb{R}^2} (x-y)^2 e^{-\lambda(x-y)} p(x)p(y)f(x)^2 dxdy \geq 0,$$

which concludes the proof.

Unfortunately, we do not know the sign of the first derivative

$$\frac{d}{d\lambda} J(\lambda) = -\int_{\mathbb{R}^2} (x-y) e^{-\lambda(x-y)} p(x)p(y)f(x)^2 dxdy,$$

which depends on p and f. But, since p and f are given, we can numerically compute J on Λ.

Example 5.1.4. (European call option) Let $f(x) := (\exp(x) - K)_+$ with some $K > 0$. We use a Lévy process $\{Y_t : t \geq 0\}$ with Lévy measure

$$\nu(dz) = C \left(\frac{e^{-G|z|}}{|z|^{1+Y}} 1(z < 0) + \frac{e^{-Mz}}{z^{1+Y}} 1(z > 0) \right) dz,$$

with $(C, G, M, Y) = (0.001, 2.0, 2.0, 1.95)$. We can prove $\Lambda = (-2, 2)$. Set $X_t := Y_t - \ln E_Q[e^{Y_t}]$ so that the discounted stock price process $S_t := e^{X_t}$ is a martingale under Q. In this example, we compute $E_Q[f(X_1)]$ under the assumption of zero risk-free rate. Then, we generate $\{x_i\}_{i \geq 1}$ with common density p, and compute the average $n^{-1} \sum_{i=1}^{n} f(x_i)$. In deep *out-of-the-money* cases (i.e., $K \gg S_0$), many of realizations $\{f(x_i)\}_{i \geq 1}$ will be zero. These non-zero realizations do not contribute to the evaluation and indeed are wasted. To avoid this problem, define an equivalent probability measure P_λ by the Esscher transform (5.1.3) with some $\lambda \in (0, 2)$ so as to tilt the density p under Q to the right. Under P_λ, we expect less realizations of zero value, and thus the convergence should be faster. Figure 5.1 draws $\{J(\lambda) - I^2 : \lambda \in \Lambda\}$ for two K's, indicating that $J(\lambda)$ are monotonically decreasing in λ. This fact indeed matches the intuition. In this specific example, we should take λ arbitrarily closer to 2 to reduce the error variance as much as possible. The simulation results in Figure

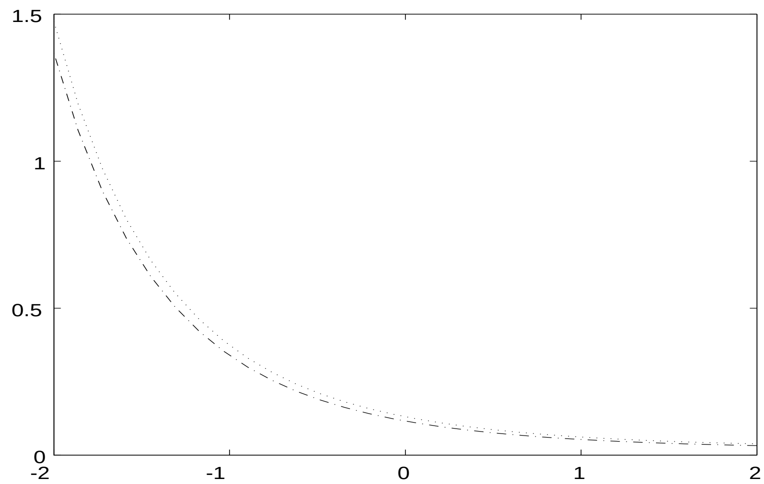


Figure 5.1: $J(\lambda) - I^2$ for $K = 1.1$ (- -) and $K = 1.05$ (-··-)

5.2 assert our conjecture. They are for $K = 1.10$ and $K = 1.05$ with the theoretical prices 0.1388 and 0.1563, respectively. Two lines indicating the theoretical price $\pm 1\%$ are also drawn. (—) is the average with $\lambda = 1.99$, (- -) with $\lambda = -1.0$, and (-··-) with $\lambda = 0.0$ (i.e., no Esscher transform). According to Figure 5.1, the variance ratios are $\mathrm{Var}_{P_{-1.0}} / \mathrm{Var}_{P_{0.0}} = 2.873$, $\mathrm{Var}_{P_{1.99}} / \mathrm{Var}_{P_{0.0}} = 0.2776$ for $K = 1.1$, and $\mathrm{Var}_{P_{-1.0}} / \mathrm{Var}_{P_{0.0}} = 3.152$, $\mathrm{Var}_{P_{1.99}} / \mathrm{Var}_{P_{0.0}} = 0.2623$ for $K = 1.05$. For the sake of clear comparison, the random sequences $\{x_i\}_{i \geq 1}$ under $P_{1.99}$, $P_{0.0}$, and $P_{-1.0}$ are generated on a common probability space, i.e., we use a common sequence $\{u_i\}_{i \geq 1}$ of iid uniform random variables in $(0, 1)$ in the distribution-function inversion method.

We have seen that in path-independent cases, the variance reduction performs well and the variance has a relatively simple structure with respect to the Esscher parameter λ. The Monte Carlo simulation is, however, essentially of no use for such cases. Indeed, as long as the density function p is known, the expectation can be numerically computed. Let us then consider more practical cases. Set a stock price

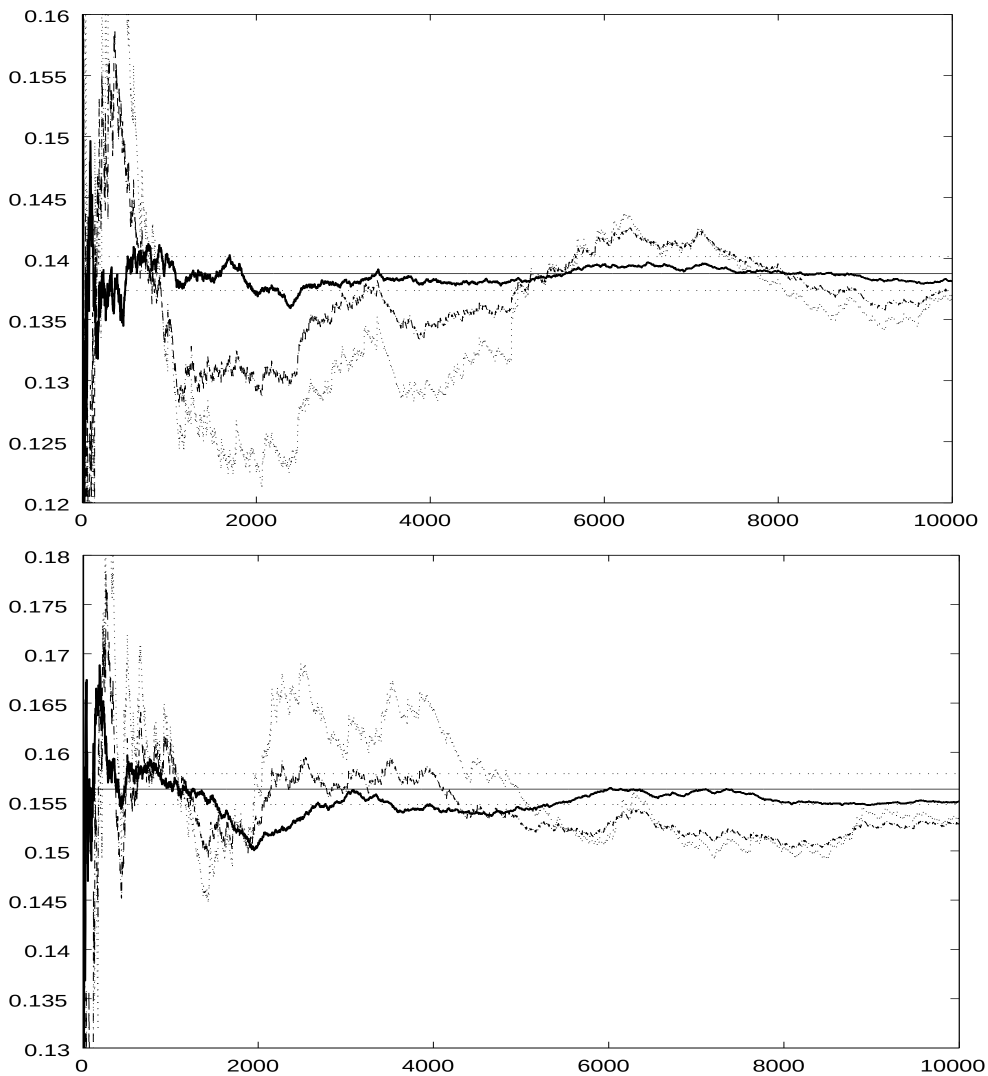


Figure 5.2: European call option with $K = 1.10$ (top), and $K = 1.05$ (bottom)

process by $S_t = S_0 \exp(X_t)$, $t \in [0, T]$. Let F be a functional from $\mathbb{D}[0, T]$ to $\mathbb{R}$. Then, $F(X) \in \mathcal{F}_T$ can be considered to be a payoff of some financial option. Typical examples are;

(i) the Asian call option; $F(X) = (\int_0^T S_t dt - K)_+$,

(ii) a look-back call option; $F(X) = (\max_{t \in [0,T]} S_t - K)_+$,

(iii) a barrier call option; $F(X) = (S_T - K_1)_+ \mathbf{1}(\max_{t \in [0,T]} S_t \geq K_2)$.

In those cases, the knowledge of the density p of X_T is clearly not enough to compute the expectation $E_Q[F]$. Indeed, we need to generate its entire paths $\{X_t^i : t \in$

$[0,T]\}_{i\geq 1}$, compute the realizations $\{F(X^i)\}_{i\geq 1}$ and take the average $n^{-1}\sum_{i=1}^{n} F(X^i)$.

Let us now apply our Esscher transform method. Suppose that the original Lévy process X is generated by $(\gamma, 0, \nu)$ under Q. Then, X is again a Lévy process under P_λ defined by $dP_\lambda/dQ|_{\mathcal{F}_T} = e^{\lambda X_T}/E_Q[e^{\lambda X_T}]$, with generating triplet $(\gamma_\lambda, 0, \nu_\lambda)$, where $\nu_\lambda(dz) = e^{\lambda z}\nu(dz)$ and

$$\gamma_\lambda = \gamma + \int_{\|z\|\leq 1} z(e^{\lambda z} - 1)\nu(dz).$$

When $\lambda > 0$, ν_λ is a right-tilted version of ν. Also $\int_{\|z\|\leq 1} z(e^{\lambda z} - 1)\nu(dz) \geq 0$ because $z(e^{\lambda z} - 1) \geq 0$ on $z \in [-1, 1]$. By those, we expect that the new Lévy process after being the Esscher transformed with some $\lambda > 0$ has paths with greater values than the original Lévy process. The expectation is computed by

$$I := E_Q[F(X)] = E_{P_\lambda}\left[\left(\frac{e^{\lambda X_T}}{E_Q[e^{\lambda X_T}]}\right)^{-1} F(X)\right].$$

If we can take λ so that

$$V(\lambda) := \mathrm{Var}_{P_\lambda}\left(\left(\frac{e^{\lambda X_T}}{E_Q[e^{\lambda X_T}]}\right)^{-1} F(X)\right) < \mathrm{Var}_Q(F(X)),$$

then the Monte Carlo simulation is expected to converge faster.

Example 5.1.5. (Asian-type call option) Consider an Asian-type call option

$$\max\left\{\frac{1}{N}\sum_{i=1}^{N} S_{\frac{i}{N}T} - K, 0\right\}.$$

In this specific example, we take $N = 5$ and $T = 1$. As in the last example, we want to reduce realizations of value zero in the Monte Carlo simulation by tilting the stock price upwards by the Esscher transform with some positive λ. We use the same CGMY process as in Example 5.1.4, and so $\Lambda = (-2, 2)$. In Figure 5.3, we give $J(\lambda) - I^2$ for both K's. This indicates that the variance decreases as $\lambda > 0$ gets greater, which matches our intuition. Note that in the case of $K = 1.1$, J is not monotone unlike in the European call option case. The simulation results are shown in Figure 5.4.

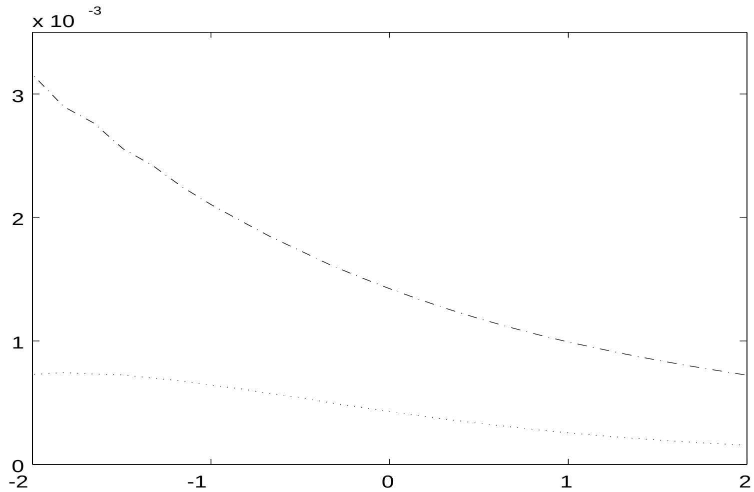


Figure 5.3: $J(\lambda) - I^2$ for $K = 1.1$ (- -) and $K = 1.05$ (-·-)

The theoretical price of the option is rather intractable. As in Example 5.1.4, (—) is the convergence with $\lambda = 1.99$, (- -) with $\lambda = -1.0$, and (-·-) with $\lambda = 0.0$, and all random sequences are generated on a common probability space. According to Figure 5.1, the variance ratios are $\mathrm{Var}_{P_{-1.0}} / \mathrm{Var}_{P_{0.0}} = 1.708$, $\mathrm{Var}_{P_{1.99}} / \mathrm{Var}_{P_{0.0}} = 0.430$ for $K = 1.1$, and $\mathrm{Var}_{P_{-1.0}} / \mathrm{Var}_{P_{0.0}} = 1.592$, $\mathrm{Var}_{P_{1.99}} / \mathrm{Var}_{P_{0.0}} = 0.577$ for $K = 1.05$. The results indicate that $\lambda = 1.99$ gives very fast convergence.

5.2 Solving Stochastic Differential Equations via Series Representations

Let $\{X_t : t \in [0, T]\}$ be an additive process in the series representation form

$$X_t := \sum_{i=1}^{\infty} \Big[H_i 1(S_i \le t) - c_i P(t) \Big], \quad t \in [0, T],$$

where $P(t) = P(S_1 \le t)$. Now, let $\{Z_t : t \in [0, T]\}$ be a solution of the stochastic differential equation

$$Z_t = Z_0 + \int_0^t f(s, Z_{s-}) dX_s + \int_0^t g(s, Z_{s-}) ds, \quad t \in [0, T], \tag{5.2.1}$$

where Z_0 is a constant a.s. The main purpose of this section is to introduce a new method to approximate the solution of this stochastic differential equation by using

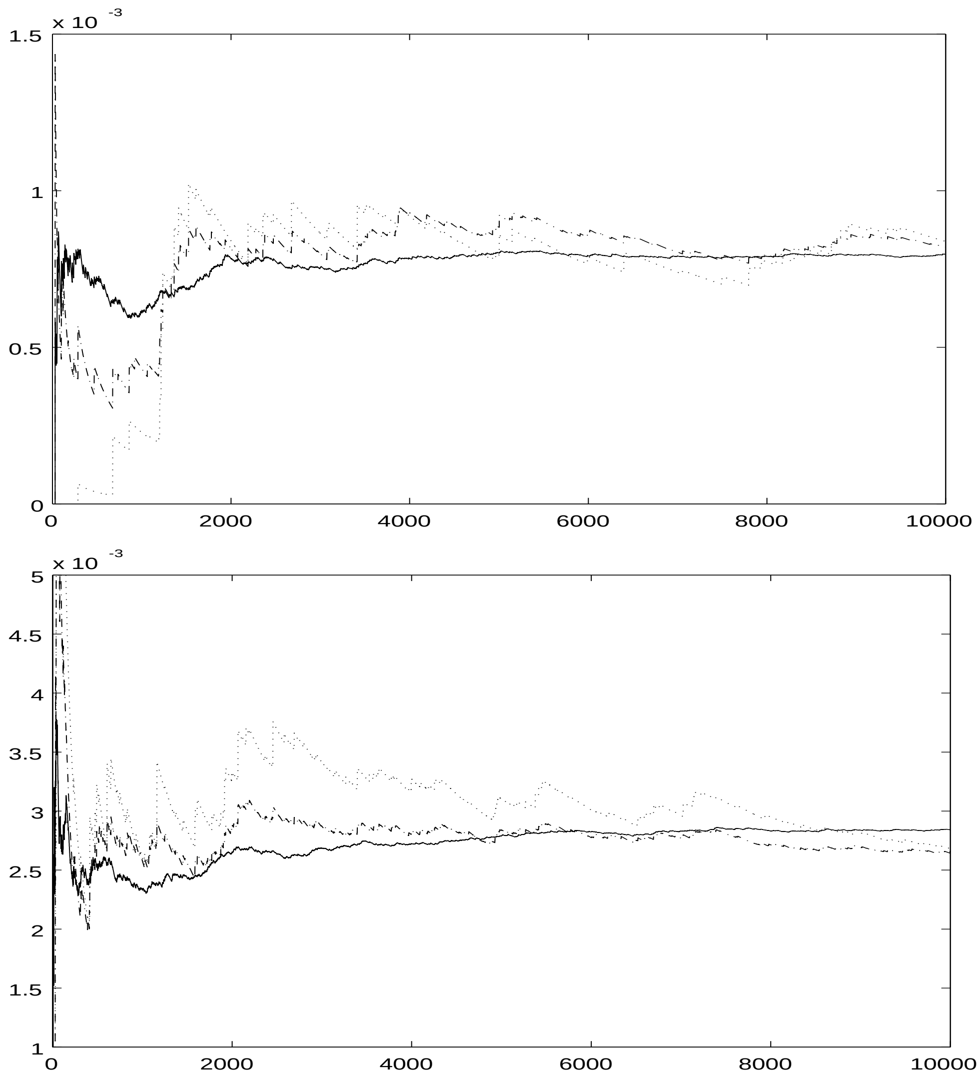


Figure 5.4: Asian-type call option with $K = 1.10$ (top), and $K = 1.05$ (bottom)

the structure of series representations.

Define a stochastic process $X^{(n)}$, consisting of a partial sum of X, by

$$X_t^{(n)} := \sum_{i=1}^{n} \Big[H_i 1(S_i \le t) - c_i P(t) \Big], \quad t \in [0,T],\ n \ge 1,$$

and set $S(n) := \{S_i\}_{i=1}^n$ with permutation $\{\theta_n(i)\}_{i=1}^n$ arranging $\{S_i\}_{i=1}^n$ in increasing order, i.e. $S_{\theta_n(1)} \le S_{\theta_n(2)} \le \cdots \le S_{\theta_n(n)}$. For convenience, put $S_{\theta_n(0)} := 0$ and $S_{\theta_n(n+1)} := T$. Moreover, define a discretized version of $X^{(n)}$ by

$$X_t^{S(n)} := \sum_{i=1}^{n} H_i 1(S_i \le t) - c(n) \sum_{i=0}^{n} P(S_{\theta_n(i)}) 1(S_{\theta_n(i)} \le t < S_{\theta_n(i+1)}),$$

where $c(n) = \sum_{i=1}^{n} c_i$. Let us define two forms of stochastic differential equations approximating (5.2.1). The first equation is based on $\{X_t^{(n)} : t \in [0,T]\}$, defined by

$$Z_t^{(n)} := Z_0 + \int_0^t f(s, Z_{s-}^{(n)}) dX_s^{(n)} + \int_0^t g(s, Z_{s-}^{(n)}) ds, \tag{5.2.2}$$

which is a kind of perturbations of the stochastic differential. The second is a discretization of (5.2.2), defined by

$$\begin{aligned} Z_t^{S(n)} \quad = \quad & Z_0 + \sum_{i=1}^{n} f(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}}) H_{S_{\theta_n(i)}} 1(S_{\theta_n(i)} \le t) \\ & - c(n) \sum_{j=1}^{n+1} P(S_{\theta_n(i)}) f(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) 1(S_{\theta_n(i)} \le t) \\ & + \sum_{i=1}^{n+1} g(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) 1(S_{\theta_n(i)} \le t). \end{aligned} \tag{5.2.3}$$

Clearly, (5.2.3) is most useful for practical purpose because the path of the solution has the form of a step function. In the following, we will show that the solution of (5.2.2) converges to the true solution as $n \to \infty$ uniformly in probability on $[0,T]$, i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} P(\sup_{t\in[0,T]} |Z_t^{S(n)} - Z_t| > \epsilon) = 0.$$

Theorem 5.2.1. *Let $\{Z_t : t \in [0,T]\}$ and $\{Z_t^{S(n)} : t \in [0,T]\}$ be solutions respectively of (5.2.1) and (5.2.3) with f and g being left continuous with right limits with respect to the first argument and Lipschitz with respect to the second argument. Then, $Z^{S(n)}$ and Z are unique solutions and semimartingales. Moreover, $Z^{S(n)}$ converges to Z uniformly in probability as $n \to \infty$.*

Proof. It is a standard fact that the conditions on f and g ensure the uniqueness of solutions and the semimartingale path property. Let $Z^{(n)}$ be defined by (5.2.2). It immediately holds that $Z^{S(n)} - Z^{(n)}$ converges to 0 uniformly probability on $[0,T]$, where $Z^{(n)}$ is defined by (5.2.2), due to the well known results on the stability of stochastic differential equations because $Z^{S(n)}$ is simply an approximation of $Z^{(n)}$

with finite differences on the random partition $S(n)$. (See, for example, Corollary (pp.213) of Protter [38].) Let us next show that $Z^{(n)} - Z$ converges to 0 uniformly in probability on $[0,T]$. But, this claim is also a consequence of the stability of stochastic differential equations because $X^{(n)}$ converges to X uniformly a.s. on $[0,T]$. (See, for example, Theorem 15 (pp.209) of Protter [38].) Therefore, the inequality

$$E\Big[\sup_{t\in[0,T]}|Z_t^{S(n)} - Z_t| \wedge 1\Big] \le E\Big[\sup_{t\in[0,T]}|Z_t^{S(n)} - Z_t^{(n)}| \wedge 1\Big] + E\Big[\sup_{t\in[0,T]}|Z_t^{(n)} - Z_t| \wedge 1\Big],$$

concludes the proof.

In view of (5.2.3), we can iteratively compute $Z^{(n)}$ as follows. For $t \in [0, S_{\theta_n(1)})$, set $Z_t^{S(n)} := Z_0$. For $t \in [S_{\theta_n(1)}, S_{\theta_n(2)})$,

$$Z_t^{S(n)} := Z_0 + H_{S_{\theta_n(1)}} f(0, Z_0) - c(n)P(S_{\theta_n(1)})f(0,Z_0)S_{\theta_n(1)} + g(0,Z_0)S_{\theta_n(1)}.$$

In general, for $t \in [S_{\theta_n(k)}, S_{\theta_n(k+1)})$, $k = 1, \ldots, n$, set

$$\begin{aligned}
Z_t^{S(n)} &= Z_0 + \sum_{i=1}^{k} H_{S_{\theta_n(i)}} f(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}}) \\
&\quad -c(n)\sum_{i=1}^{k} P(S_{\theta_n(i)}) f(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \\
&\quad +\sum_{i=1}^{k} g(S_{\theta_n(i-1)}, Z^{S(n)}_{S_{\theta_n(i-1)}})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \\
&= Z^{S(n)}_{S_{\theta_n(k-1)}} + H_{S_{\theta_n(k)}} f(S_{\theta_n(k-1)}, Z^{S(n)}_{S_{\theta_n(k-1)}}) \\
&\quad -c(n)P(S_{\theta_n(k)}) f(S_{\theta_n(k-1)}, Z^{S(n)}_{S_{\theta_n(k-1)}})(S_{\theta_n(k)} - S_{\theta_n(k-1)}) \\
&\quad +g(S_{\theta_n(k-1)}, Z^{S(n)}_{S_{\theta_n(k-1)}})(S_{\theta_n(k)} - S_{\theta_n(k-1)}),
\end{aligned}$$

and

$$Z_T^{S(n)} = Z^{S(n)}_{S_{\theta_n(n)}} - c(n)f(S_{\theta_n(n)}, Z^{S(n)}_{S_{\theta_n(n)}})(T - S_{\theta_n(n)}) + g(S_{\theta_n(n)}, Z^{S(n)}_{S_{\theta_n(n)}})(T - S_{\theta_n(n)}).$$

Our method simultaneously resolves two problems of the standard Euler scheme.

In the standard Euler scheme, the solution Z of (5.2.1) is approximated by the recursive solution

$$\widetilde{Z}_{(i+1)\Delta} = \widetilde{Z}_{i\Delta} + f(i\Delta, \widetilde{Z}_{i\Delta})(X_{(i+1)\Delta} - X_{i\Delta}) + g(i\Delta, \widetilde{Z}_{i\Delta})\Delta,$$

where $\Delta := T/n$. The first problem issues on the generation of the sequence of independent random variables $\{X_{(i+1)\Delta} - X_{i\Delta}\}_{i=0}^{n}$. Clearly, it is just a sequence of iid random variables if X is a Lévy process. But, it is not if X is an general additive process and thus one need to obtain different distribution for each elements. In non-Gaussian cases, this requires one to perform the Fourier inversion for each time interval, which is computationally very expensive. Next, to achieve higher precision of the approximation, we let $\Delta \to 0$ by the standard fact that $\widetilde{Z} \to Z$ uniformly in probability as $\Delta \to 0$. The second problem arises here. In general, the Fourier inversion gives a very distorted density as $\Delta \to 0$. On the other hand, all our method needs for computation is only *one* form of series representations. In particular, our method does not need the Fourier inversion. By Theorem 5.2.1, higher precision of the law of the solution Z can be achieved simply by increasing the number of summands of the series representation.

Example 5.2.2. (Stochastic exponential) Let $\{Z_t : t \in [0,T]\}$ be the strong unique solution of the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-}dX_s$, and $\{Z_t^{S(n)}\}$ a solution of

$$\begin{aligned} Z_t^{S(n)} &= 1 + \sum_{i=1}^{n} Z^{S(n)}_{S_{\theta_n(i-1)}} H_{\theta_n(i)} 1(S_{\theta_n(i)} \le t) \\ &\quad - c(n) \sum_{i=1}^{n+1} Z^{S(n)}_{S_{\theta_n(i-1)}} P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) 1(S_{\theta_n(i)} \le t). \end{aligned}$$

By Theorem 5.2.1, $Z^{S(n)} \to Z$ uniformly in probability as $n \to \infty$. On the other hand, we can derive by induction that

$$Z^{S(n)}_{S_{\theta_n(k)}} = \prod_{i \le k} \left[1 + H_{\theta_n(i)} - c(n)P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)})\right].$$

It is well known that this converges to the true solution uniformly in probability. (See, for example, Theorem 17 (pp.214) of Protter [38].)

Example 5.2.3. (Lévy Bridge) Let $\{Z_t : t \in [0,1]\}$ be a solution of the stochastic differential equation

$$Z_t = a + \int_0^t \frac{b - Z_s}{1-s} ds + X_t, \quad t \in [0,1],\ Z_0 = a,$$

for some $a, b \in \mathbb{R}$. It is known that Z is uniquely given by

$$Z_t = a(1-t) + bt + (1-t)\int_0^t \frac{dX_s}{1-s}, \quad t \in [0,1], \tag{5.2.4}$$

with $Z_1 = b$ a.s. In our framework, this is formulated by $f \equiv 1$ and $g(s,z) = (b-z)/(1-s)$;

$$\begin{aligned} Z_t^{S(n)} = a &+ \sum_{i=1}^{n+1} \frac{b - Z^{S(n)}_{S_{\theta_n(i-1)}}}{1 - S_{\theta_n(i-1)}} (S_{\theta_n(i)} - S_{\theta_n(i-1)}) 1(S_{\theta_n(i)} \le t) \\ &+ \sum_{i=1}^{n} \left[H_{\theta_n(i)} - c(n) P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \right] 1(S_{\theta_n(i)} \le t). \end{aligned}$$

We can prove by induction that this is equivalent to

$$\begin{aligned} Z^{S(n)}_{S_{\theta_n(k)}} &= a(1 - S_{\theta_n(k)}) + bS_{\theta_n(k)} \\ &+ (1 - S_{\theta_n(k)}) \sum_{i=1}^{n} \frac{H_{\theta_n(i)} - c(n)P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)})}{1 - S_{\theta_n(i)}}. \end{aligned} \tag{5.2.5}$$

Notice that the closed form solution (5.2.4) can be recovered from (5.2.5).

Example 5.2.4. (Process of Ornstein-Uhlenbeck type) Let $\{Z_t : t \ge 0\}$ be a solution of the stochastic differential equation

$$Z_t = a + X_t - b\int_0^t Z_s ds, \quad t \in [0,1], \quad Z_0 = a,$$

for some $a, b \in \mathbb{R}$. Then, Z is uniquely given by

$$Z_t = e^{-bt} a + \int_0^t e^{-b(t-s)} dX_s, \quad t \in [0,1]. \tag{5.2.6}$$

In our framework, this is given by $f \equiv 1$ and $g(s,z) = bz$;

$$Z_t^{S(n)} = a + \sum_{i=1}^{n} \left[H_{\theta_n(i)} - c(n) P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \right] 1(S_i \le t)$$
$$-b \sum_{i=1}^{n+1} Z^{(n)}_{S_{\theta_n(i-1)}} (S_{\theta_n(i)} - S_{\theta_n(i-1)}) 1(S_{\theta_n(i)} \le t).$$

By induction, we obtain

$$Z^{S(n)}_{S_{\theta_n(k)}} = a \prod_{i=1}^{k} (1 - b(S_{\theta_n(i)} - S_{\theta_n(i-1)}))$$
$$+ \sum_{i=1}^{k} \left[H_{\theta_n(i)} - c(n) P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \right]$$
$$\prod_{j=1}^{k-i} (1 - b(S_{\theta_n(k-j+1)} - S_{\theta_n(k-j)})),$$

and

$$Z_1^{S(n)} = Z^{S(n)}_{S_{\theta_n(n)}} + a \prod_{i=1}^{n+1} (1 - b(S_{\theta_n(i)} - S_{\theta_n(i-1)})).$$

Since $\lim_{n\to\infty} \sup_{0 \le k \le n+1} (S_{\theta_n(k)} - S_{\theta_n(k-1)}) = 0$ a.s., we get

$$Z^{S(n)}_{S_{\theta_n(k)}} \sim a \prod_{i=1}^{k} e^{-b(S_{\theta_n(i)} - S_{\theta_n(i-1)})}$$
$$+ \sum_{i=1}^{k} \left[H_{\theta_n(i)} - c(n) P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \right] \prod_{j=1}^{k-i} e^{-b(S_{\theta_n(k-j+1)} - S_{\theta_n(k-j)})}$$
$$= a e^{-b S_{\theta_n(k)}} + \sum_{i=1}^{k} e^{-b(S_{\theta_n(k)} - S_{\theta_n(i)})} \left[H_{\theta_n(i)} - c(n) P(S_{\theta_n(i-1)})(S_{\theta_n(i)} - S_{\theta_n(i-1)}) \right],$$

which yields (5.2.6).

In Figure 5.5, we give typical sample paths of the Lévy bridge and a process of Ornstein-Uhlenbeck type, simulated by our series representation method. The background driving Lévy process is set to be a CGMY process with $(C, G, M, Y) = (0.001, 1.0, 1.0, 1.75)$. We set $a = b = 0.1$. (—) indicates the path with $n = 5000$ (the number of the summands), (-·-) with $n = 3000$, and (- -) with $n = 500$. Apparently, sample paths converge as n gets greater.

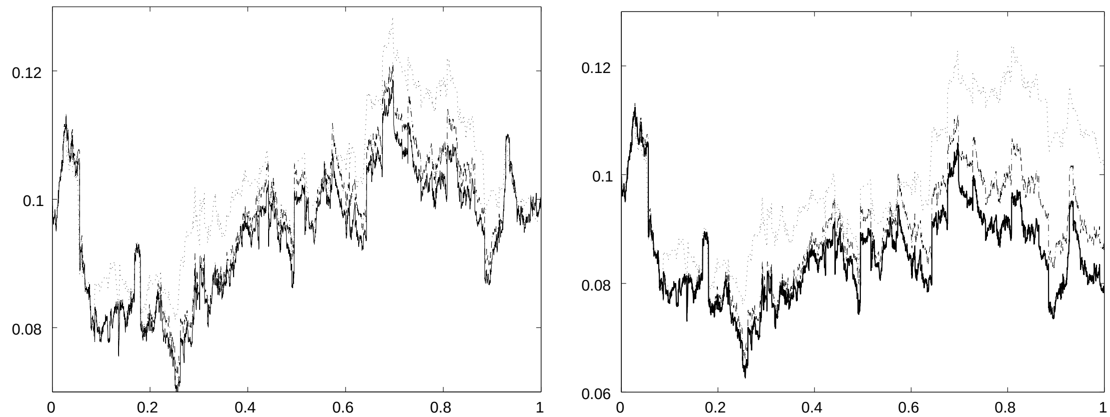


Figure 5.5: Typical sample paths of the Lévy bridge (left) and a process of Ornstein-Uhlenbeck type (right)

5.3 Minimal Variance Hedging

The field of mathematical finance has been growing since Black and Scholes [7] published their famous paper in which they derived explicit valuation formulas of European option prices whose underlying asset price $\{S_t : t \geq 0\}$ is modeled as

$$dS_t = rS_{t-}dt + \sigma S_{t-}dB_t,$$

where $r \geq 0$ is a risk-free interest rate, $\sigma > 0$ is called the volatility, and $\{B_t : t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}$. This framework is called the *Black-Scholes model*. Moreover, the concept of the arbitrage-free hedging was mathematically formulated based upon the martingale theory by Harrison and Kreps [15] and Harrison and Pliska [16]. Let us first review some basic facts. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a natural filtration generated by $\{S_t : t \in [0,T]\}$ and let $\xi \in \mathcal{F}_T$. The random variable ξ usually represents the payoff of financial derivatives and is called a contingent claim. Then, "a contingent claim ξ is hedgable" is represented by

$$\xi = E[\xi] + \int_0^T \theta_t dS_t \quad a.s., \tag{5.3.1}$$

where $E[\xi]$ is a fair option price and $\{\theta_t : t \in [0,T]\}$ is predictable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Here, θ_t represents a *portfolio* at time t, i.e. the number of units of the underlying asset to be held at time t. The θ satisfying (5.3.1) is called an *admissible hedging portfolio*. The market in which every contingent claim is hedgable is said to be *complete,* otherwise *incomplete*. The completeness corresponds to the uniqueness of the equivalent martingale measure. It is known that the market is complete if every asset price process is driven either by a Brownian motion or a Poisson process. However, general additive processes induce imcomplete markets. This fact will be justified in Proposition 5.3.1.

In the incomplete market cases, define a replication ξ_θ of ξ by

$$\xi_\theta := E[\xi] + \int_0^T \theta_t dS_t. \tag{5.3.2}$$

We note that $\xi_\theta \neq \xi$ in general. It is then a natural question how closely one can replicate a contingent claim ξ by ξ_θ in (5.3.2). The closeness is usually measured in terms of the variance of the difference. Letting $\mathcal{A}$ be a collection of all admissible portfolios and ξ be a square integrable contingent claim, this problem reduces to finding a portfolio φ satisfying

$$E\left[(\xi - \xi_\varphi)^2\right] = \inf_{\theta \in \mathcal{A}} E\left[(\xi - \xi_\theta)^2\right] = \inf_{\theta \in \mathcal{A}} E\left[\left(\xi - E[\xi] - \int_0^T \theta_t dS_t\right)^2\right].$$

The portfolio φ is usually called the *minimal variance hedging portfolio.*

The main purpose of this section is to verify the applicability of Lévy processes and additive processes as replacements of Brownian motions in the Black-Scholes model, in terms of the minimal variance hedging. Let us first give the main result. Here, $\{G_t : t \in [0,T]\}$ is a centered Gaussian process in $\mathbb{R}$ with independent increments, $G_0 = 0$, and $E[G_t^2] = \int_0^t \sigma^2(s)ds$ for some nonnegative continuous function σ. Moreover, ς is a Poisson random measure whose intensity measure ϱ can be decomposed as $\varrho(dz, ds) = \nu(dz)y(s)ds$ where ν is a Lévy measure satisfying $\int_{\mathbb{R}_0}(e^z - 1)^2\nu(dz) < \infty$ and y is some nonnegative function.

Proposition 5.3.1. *Assume that the risk-free interest rate is 0. Define an asset price process* $\{S_t : t \in [0,T]\}$ *by*

$$dS_t = S_{t-}\left(dG_t + \int_{\mathbb{R}_0}(e^z - 1)(\varsigma - \varrho)(dz, dt)\right), \tag{5.3.3}$$

Then, the minimal variance portfolio φ *of a contingent claim* $\xi \in \mathcal{F}_T$ *is given by*

$$\varphi_t = \frac{\sigma^2(t)E[D_t\xi|\mathcal{F}_{t-}] + y(t)\int_{\mathbb{R}_0}(e^z-1)E[D_{z,t}\xi|\mathcal{F}_{t-}]\nu(dz)}{S_{t-}\left(\sigma^2(t) + y(t)\int_{\mathbb{R}_0}(e^z-1)^2\nu(dz)\right)} \tag{5.3.4}$$

where D_t *and* $D_{z,t}$ *are the Malliavin derivative operators defined by (1.5.3). Moreover, the* φ *gives the zero variance if and only if either* $\nu \equiv 0$*, or* $\sigma \equiv 0$ *and* $\nu = \delta_a$ *for some* $a \in \mathbb{R}_0$.

Proof. Clearly, $\{S_t : t \in [0,T]\}$ is a martingale and

$$\xi_\theta = E[\xi] + \int_0^T \theta_s S_{s-} dG_s + \int_0^T \int_{\mathbb{R}_0} \theta_s S_{s-}(e^z - 1)(\varsigma - \varrho)(dz, ds).$$

By the Clark-Ocone formula (1.5.2), we have

$$\xi = E[\xi] + \int_0^T E[D_s\xi|\mathcal{F}_{s-}]dG_s + \int_0^T \int_{\mathbb{R}_0} E[D_{z,s}\xi|\mathcal{F}_{s-}](\varsigma - \varrho)(dz, ds),$$

and thus we get

$$\begin{aligned}\xi - \xi_\theta = &\int_0^T (E[D_s\xi|\mathcal{F}_{s-}] - \theta_s S_{s-})dG_s \\ &+ \int_0^T \int_{\mathbb{R}_0} (E[D_{z,s}\xi|\mathcal{F}_{s-}] - \theta_s S_{s-}(e^z-1))(\varsigma - \varrho)(dz, ds).\end{aligned}$$

Therefore, the Itô isometry (1.5.1) gives

$$\begin{aligned}E\left[(\xi - \xi_\theta)^2\right] &= E\Bigg[\int_0^T \Big(\sigma^2(s)(E[D_s\xi|\mathcal{F}_{s-}] - \theta_s S_{s-})^2 \\ &\quad + \int_{\mathbb{R}_0}(E[D_{z,s}\xi|\mathcal{F}_{s-}] - \theta_s S_{s-}(e^z-1))^2\nu(dz)y(s)\Big)ds\Bigg],\end{aligned} \tag{5.3.5}$$

which is minimized by (5.3.4). The second claim is immediate from (5.3.5).

Remark 5.3.2. This result is a modification of Benth et al [6]. In their framework, the asset price process is modeled as a Lévy process

$$S_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z(\mu - \nu)(dz, ds),$$

which easily becomes negative as in the setting of Bachelier [3]. On the other hand, by the Itô formula, our formulation (5.3.3) satisfies

$$\begin{aligned} S_t = S_0 \exp\Bigg[& -\frac{1}{2}\int_0^t \sigma^2(s)ds \\ & - \int_0^t \int_{\mathbb{R}_0} (e^z - 1 - z)\varrho(dz, ds) + G_t + \int_0^t \int_{\mathbb{R}_0} z(\varsigma - \varrho)(dz, ds)\Bigg]. \end{aligned} \tag{5.3.6}$$

We also note that $\nu \equiv 0$ induces purely Gaussian, and $\sigma \equiv 0$ and $\nu = \delta_a$ induces a Poisson process.

The following corollary gives the minimal variance hedging portfolio for the European call option.

Corollary 5.3.3. *If $\xi = f(S_T) := (S_T - K)^+$ for some $K > 0$, then the minimal variance portfolio of ξ is given by*

$$\varphi_t = \frac{\sigma^2(t)E[S_T 1_{S_T \geq K}|\mathcal{F}_{t-}] + y(t)\int_{\mathbb{R}_0}(e^z - 1)E[(e^z S_T - K)^+ - (S_T - K)^+|\mathcal{F}_{t-}]\nu(dz)}{S_{t-}\left(\sigma^2(t) + y(t)\int_{\mathbb{R}_0}(e^z - 1)^2\nu(dz)\right)}. \tag{5.3.7}$$

Proof. It suffices to consider the Gaussian component and the pure jump component separately. Note that f is not differentiable at $x = K$. For the Gaussian component, the nondifferentiability of f at $x = K$ has to be taken care of. Following Øksendal [36], define $f_n \in C^1$ such that $f_n(x) = f(x)$ for $|x - K| \geq n^{-1}$ and $0 \leq f_n' \leq 1$. For $t \in [0, T]$ and $z \in \mathbb{R}_0$, we get $D_t f(S_T) = \lim_{n\to\infty} D_t f_n(S_T)$ and $D_{z,t} f(S_T) = \lim_{n\to\infty} D_{z,t} f_n(S_T)$. Then, for the Gaussian component, by (1.5.3), $D_t f(S_T) = \lim_{n\to\infty} f_n'(S_T) D_t S_T = \sigma S_T 1_{S_T \geq K}$. For the pure jump component, clearly, $D_{z,t} f(S_T) = f(e^z S_T) - f(S_T)$. The rest is straightforward.

By discretize the asset price (5.3.3) along with the equidistant partition of $[0,T]$; $t_i = i(T/n)$ for $i = 0, 1, \ldots, n$, i.e.,

$$Z_i := \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} = (G_{t_{i+1}} - G_{t_i}) + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}_0} (e^z - 1)(\varsigma - \varrho)(dz, ds),$$

we get $E[Z_i | \mathcal{F}_{t_i}] = 0$ and

$$\mathrm{Var}(Z_i | \mathcal{F}_{t_i}) = \int_{t_i}^{t_{i+1}} \sigma^2(s) ds + \int_{t_i}^{t_{i+1}} y(s) ds \int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz).$$

If the asset price were driven by a Lévy process, then $\{Z_i\}_{i \geq 1}$ would look like a stationary noise. In Figure 5.6, we plot $\{Z_i\}_{i=1}^{1231}$ with the equidistant partition of the stock price of TOYOTA 4/15/1998-4/15/2003 (1232 business days). An estimation of the variances $\{\mathrm{Var}(Z_i|\mathcal{F}_{t_i})\}_{i \leq n}$ is computed by $\sum_{j=0}^{n} Z_{i-j} K_n(j)$ where $K_n(j) = c_n(1 - (j/n)^2)$ with c_n a normalizing constant so that $\sum_{j=0}^{n} K_n(j) = 1$. Here, we set $n = 10$. The time-inhomogeneity is apparent.

Let us present numerical results on the minimal variance hedging for the European call option $\xi = (S_T - K)^+$. Underlying assets are stock prices of TOYOTA, SONY, HONDA, YAMAHA, all in Tokyo stock exchange. For simplicity, we consider short maturities of $T = 15/246$, $30/246$ and $60/246$ (years), each of which maturity date is 4/15/2003. (One year consists of 246 business days.) Moreover, the risk-free rate in the Japanese market is around 0.005 on the annual basis, which is negligibly small. In this setting, we compare the performance of three models;

(i) the (standard) Black-Scholes model (BS); $\nu \equiv 0$ and $\sigma \equiv C > 0$,

(ii) a Lévy process model (LP); $\sigma \equiv 0$ and $y \equiv 1$,

(iii) an additive process model (AP); $\sigma \equiv 0$.

The hedging procedure for BS is a standard topic, so we do not touch that. For LP, the asset price (5.3.3) is originally modeled as a martingale. Under $\sigma \equiv 0$, we estimate the Lévy measure ν (of tempered stable distributions) from daily data 4/15/1998-(4/15$-n$)/2003 as in Section 2.6. The computation of $E[\xi]$ is straightforward and the

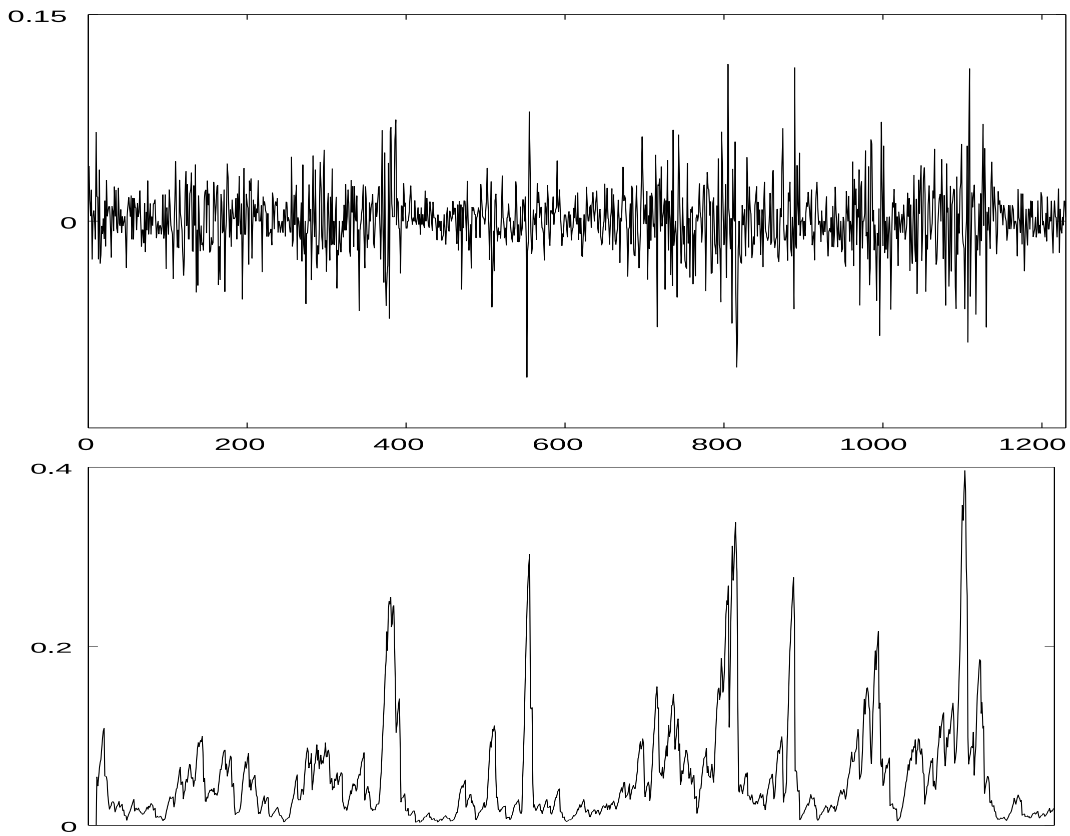


Figure 5.6: Time series $\{Z_i\}$ (top), and estimated variance $\{\mathrm{Var}(Z_i|\mathcal{F}_{t_i})\}$ for $n = 10$

minimal variance hedging portfolio is computed by (5.3.7) with $\sigma \equiv 0$. For AP, we set $\sigma \equiv 0$ and estimate the timer y from historical data. We will use the same option price and the Lévy measure as those in LP, and will update the timer from historical data step by step. The results are shown in Table 5.1, 5.2, 5.3 and 5.4. Observe first that BS evaluates the option price $E[\xi]$ higher than LP, especially near at-the-money position. This fact is illustrated in Figure 5.7, where the European call option price difference in Japanese yen is drawn against the stock-strike ratio S_0/K for 3-day, 5-day and 10-day maturities. Here, we set $S_0 = 1$. The BS prices are too high near at-the-money, where most of the derivatives are traded. At a very deep position of in-the-money and out-of-the-money, option price is model independent. Let us now return to the tables. Due to the high option price near at-the-money, the replication

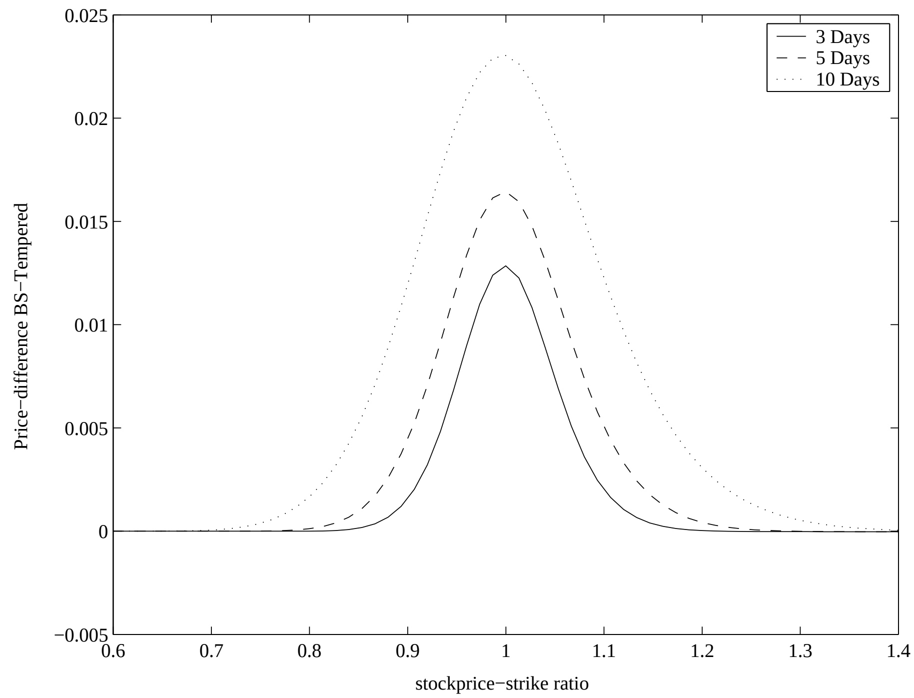


Figure 5.7: Price difference; Black-Scholes model minus tempered stable process model

precision for at-the-money positions is in general much lower than the others. This is, however, improved by LP. Notice that LP also better the precision for out-of-the-money and in-the-money positions. These observations assert the applicability of Lévy processes as a background driving process for asset prices. Moreover, AP further improves the results by LP. The improvement is evident especially in longer maturity cases. We also conjecture that the applicability of AP turns out to be even more apparent for options with longer maturity. It is safe to conclude that asset prices should not be assumed to be time-homogeneous.

5.4 Concluding Remarks

The concept of the variance reduction by the change of measure has been discussed in the framework of the important sampling. (See, for example, Andersen [1].) An extension of our results via more general density transformations may be a future

Table 5.1: Performance of minimal variance hedging portfolios : TOYOTA

		15-day			30-day			60-day		
K		2873	3380	3887	2967	3490	4014	3009	3540	4071
ξ		637	130	0	543	20	0	501	0	0
AP	ξ_φ	637.81	140.60	9.93	547.77	64.66	4.59	505.74	25.85	33.36
	$E[\xi]$	463.80	79.62	7.03	480.71	113.18	27.80	523.05	171.53	73.67
	$\int \varphi dS$	174.01	60.98	2.90	67.06	-48.52	-23.21	-17.31	-145.68	-40.31
LP	ξ_φ	637.93	140.93	10.11	551.30	76.97	8.54	507.12	67.66	42.66
	$E[\xi]$	463.80	79.62	7.03	480.71	113.18	27.80	523.05	171.53	73.67
	$\int \varphi dS$	174.13	61.31	3.08	70.59	-36.21	-19.26	-15.93	-103.87	-31.01
BS	ξ_φ	638.00	199.73	13.32	558.71	173.15	13.45	568.96	161.98	53.92
	$E[\xi]$	510.35	122.74	8.00	541.05	177.20	30.80	587.70	252.28	82.76
	$\int \varphi dS$	127.65	76.99	5.32	17.66	-4.05	-17.35	-18.74	-90.30	-28.84

Table 5.2: Performance of minimal variance hedging portfolios : SONY

		15-day			30-day			60-day		
K		4527	5325	6124	5011	5895	6780	5270	6200	7130
ξ		1138	340	0	654	0	0	395	0	0
AP	ξ_φ	1134.49	369.78	35.39	653.90	32.73	6.02	467.51	29.67	51.98
	$E[\xi]$	695.48	169.19	18.36	892.11	293.50	69.13	919.34	347.78	166.03
	$\int \varphi dS$	439.01	200.59	17.03	-238.21	-260.77	-63.11	-451.83	-318.11	-114.05
LP	ξ_φ	1133.94	372.54	39.26	676.35	62.33	10.35	531.10	65.19	73.42
	$E[\xi]$	695.48	169.19	18.36	892.11	293.50	69.13	919.34	347.78	166.03
	$\int \varphi dS$	438.46	203.35	20.92	-215.76	-231.17	-58.78	-388.24	-282.59	-92.61
BS	ξ_φ	1131.30	424.01	56.97	698.39	125.64	19.32	592.93	134.30	80.56
	$E[\xi]$	808.70	218.93	21.72	932.16	339.55	76.78	1069.20	501.43	195.95
	$\int \varphi dS$	322.58	205.08	35.25	-233.78	-213.92	-57.46	-476.23	-367.13	-115.40

research topic.

In Section 5.2, we have shown that stochastic differential equations can be solved via the series representations. A similar technique may also be employed for infinitely divisible processes without independent increments, for example, semimartingale modifications of fractional tempered stable motions.

In Section 5.3, the concept of the minimal variance hedging is used primarily to assert the applicability of Lévy processes and additive processes as a replacement of the Brownian motion for financial modeling. It would also be interesting to pursue

Table 5.3: Performance of minimal variance hedging portfolios : HONDA

		15-day			30-day			60-day		
K		4004	4710	5417	3995	4700	5405	4361	5130	5900
ξ		606	0	0	615	0	0	249	0	0
AP	ξ_φ	609.46	36.35	9.2	626.50	29.40	24.82	320.72	24.19	33.21
	$E[\xi]$	645.79	128.39	18.18	689.55	181.26	57.31	858.82	282.83	156.75
	$\int \varphi dS$	-36.33	-92.04	-8.98	-63.05	-151.86	-32.49	-538.10	-258.64	-123.54
LP	ξ_φ	609.84	38.51	10.32	636.57	53.97	30.37	412.39	68.80	52.57
	$E[\xi]$	645.79	128.39	18.18	689.55	181.26	57.31	858.82	282.83	156.75
	$\int \varphi dS$	-35.95	-89.88	-7.86	-52.98	-127.29	-26.94	-446.43	-214.03	-103.18
BS	ξ_φ	611.49	100.98	12.74	651.79	163.75	50.91	462.84	187.53	63.04
	$E[\xi]$	715.95	195.60	19.98	744.71	273.48	63.16	887.24	419.13	165.79
	$\int \varphi dS$	-104.46	-94.61	-7.25	-92.92	-109.73	-12.25	-424.40	-231.59	-102.76

Table 5.4: Performance of minimal variance hedging portfolios : YAMAHA

		15-day			30-day			60-day		
K		1011	1189	1368	1131	1330	1530	1139	1340	1541
ξ		217	39	0	97	0	0	89	0	0
AP	ξ_φ	219.98	47.98	13.14	96.55	10.69	0.43	96.15	12.41	3.33
	$E[\xi]$	180.97	37.08	7.13	182.89	61.62	22.36	219.88	85.03	50.02
	$\int \varphi dS$	39.01	10.90	6.01	-86.34	-50.93	-21.93	-123.73	-72.62	-46.69
LP	ξ_φ	220.07	48.01	13.83	95.50	12.39	0.42	116.72	17.35	8.32
	$E[\xi]$	180.97	37.08	7.13	182.89	61.62	22.36	219.88	85.03	50.02
	$\int \varphi dS$	39.10	10.93	6.70	-87.39	-49.23	-21.94	-103.16	-67.68	-41.70
BS	ξ_φ	221.68	70.58	14.59	125.62	35.99	4.32	131.18	21.66	8.64
	$E[\xi]$	182.36	55.51	7.80	215.70	87.11	24.86	241.82	123.21	55.81
	$\int \varphi dS$	39.32	15.06	6.79	-90.08	-51.11	-20.54	-110.64	-101.55	-47.17

higher precision of the replication. First, the discretization error induced by the use of daily data could be reduced by using intraday data, e.g. hourly data, if available. It is worth mentioning that the use of intraday data might also improve the estimation of the Lévy measure and the timer. Secondly, the martingale assumption on the asset price processes (5.3.3) turned out to be not too restrictive in the numerical experiments. With certainty, the results will be polished if we start with more general model, which requires a complex extension of Theorem 5.3.1. In practice, however, one will have to trade off modeling precision and computational tractability.

References

[1] L. Andersen, "Efficient techniques for simulation of interest rate models involving non-linear stochastic differential equations," *Working paper*, General Re Financial Products, New York, 1995.

[2] S. Asmussen and J. Rosiński, "Approximation of small jumps of Lévy processes with a view towards simulation," *J. Appl. Prob.*, vol. 38, pp. 482-493, 2001.

[3] L. Bachelier, "Théorie de la spéculation," *Ann. Sci. Ecole Norm. Sup.*, vol. 17, pp. 21-86, 1900.

[4] O. Barndorff-Nielsen, M. Maejima, and K. Sato, "Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation," *Preprint*, 2004.

[5] A. Benassi, S. Cohen, and J. Istas, "On roughness indices for fractional fields," *Bernoulli*, vol. 10, pp. 357-373, 2004.

[6] F. E. Benth, G. DiNunno, A. Løkka, B. Øksendal, and F. Proske, "Explicit representation of the minimal variance portfolio in market driven by Lévy processes," *Math. Finance*, vol. 13, pp. 54-72, 2003.

[7] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *J. Polit. Econ.*, vol. 81, pp. 637-659, 1973.

[8] J. Breton, C. Houdré, and N. Privault, "Dimension free and infinite variance tail estimates on Poisson space," *Preprint*, 2004.

[9] P. Carr, H. Geman, D. Madan, and M. Yor, "The fine structure of asset returns: an empirical investigation," *J. Business*, vol. 75, pp. 305-332, 2002.

[10] S. Cohen and J. Istas, "An universal estimator of local self-similarity," *Preprint*, Universitè Paul Sabatier, 2002.

[11] L. Decreusefond and A. Üstunel, "Stochastic analysis of the fractional Brownian motion," *Potential Analysis*, vol. 10, pp. 177-214, 1997.

[12] G. DiNunno, T. Meyer-Brandis, B. Øksendal, and F. Proske, "Malliavin calculus for Lévy processes," *Preprint Series in Pure Mathematics,* no. 16, University of Oslo, 2003.

[13] T. S. Ferguson and M. J. Klass, "A representation of independent increment processes without Gaussian components," *Ann. Math. Statist.*, vol. 43, pp. 1634-1643, 1972.

[14] J. E. Figueroa-Lopez, "Nonparametric estimation of Lévy processes with a view towards mathematical finance," *Ph.D. dissertation*, School of Mathematics, Georgia institute of technology, 2004.

[15] J. M. Harrison and D. Kreps, "Martingales and arbitrage in multiperiod securities markets," *J. Econ. Theory*, vol. 20, pp. 281-408, 1979.

[16] J. M. Harrison and S. Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stoch. Proc. Appl.*, vol. 11, pp. 215-260, 1981.

[17] C. Houdré and J. Villa, "An example of infinite dimensional quasi-helix," *Contemporary Mathematics*, vol. 336, pp. 195-201, 2004.

[18] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α-stable Stochastic Processes*, Marcel Dekker, Inc., New York, 1994.

[19] O. Kallenberg, "Series of random processes without discontinuities of the second kind," *Ann. Probab.*, vol. 2, pp. 729-737, 1974.

[20] O. Kallenberg, *Foundations of Modern Probability.* (2nd ed.) Springer, New York, 2001.

[21] A. Khintchine, "Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze," *Mat. Sbornik*, vol. 44, pp. 79-119, 1937.

[22] A. N. Kolmogorov, "Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen raum," *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS (N.S.),* vol. 26, pp. 115-118, 1940.

[23] I. Koponen, "Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process," *Phys. Rev. E*, 52, pp. 1197-1199, 1995.

[24] V. M. Kruglov, "A note on infinitely divisible distributions," *Theory Probab. Appl.*, vol. 15, pp. 319-324, 1970.

[25] R. LePage, "Multidimensional infinitely divisible variables and processes II," *Lecture Notes in Mathematics*, no. 860, Springer-Verlag, Berlin, New York, Heidelberg, pp. 279-284, 1980.

[26] S. J. Lin, "Stochastic analysis of fractional Brownian motions," *Stochastics Stochastics Rep.*, vol. 55, pp. 121-140, 1995.

[27] A. Løkka, "Martingale representation, chaos expansion and Clark-Ocone formulas," *Preprint,* no. 22, Center for Mathematical Physics and Stochastics, University of Aarhus, 1999.

[28] B. Mandelbrot, *Fractals and Scaling in Finance,* Springer-Verlag, New York, 1997.

[29] B. Mandelbrot and J. W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Rev.*, vol. 10, pp. 422-437, 1968.

[30] R. N. Mantegna and H. E. Stanley, "Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flight," *Phys. Rev. Letters*, vol. 73, pp. 2946-2949, 1994.

[31] G. Maruyama, "Infinitely divisible processes," *Theory Probab. Appl.*, vol. 15, pp. 1-22, 1970.

[32] E. A. Novikov, "Infinitely divisible distributions in turbulence," *Phys. Rev. E*, vol. 50, pp. 3303-3305, 1989.

[33] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer-Verlag, 1995.

[34] D. Nualart, "Stochastic integration with respect to fractional Brownian motion and applications," *Contemporary Mathematics,* 2004.

[35] D. Nualart and W. Schoutens, "Chaotic and predictable representations for Lévy processes," *Stoch. Proc. Appl.*, vol. 90, pp. 109-122, 2000.

[36] B. Øksendal, "An introduction to Malliavin calculus with applications to economics," *Working paper*, no. 3, Norweigian School of Economics and Business Administration, 1996.

[37] S. Orey, "On continuity properties of infinitely divisible distribution functions," *Ann. Math. Statist.*, vol. 39, pp. 936-937, 1968.

[38] P. Protter, *Stochastic integration and differential equation.* Springer-Verlag, 1990.

[39] J. Rosiński, "On a class of infinitely divisible processes represented as mixtures of Gaussian processes," *In Stable Processes and Related Topics. Progress in Probability,* vol. 25, Birkhäuser, pp. 27-41, 1991.

[40] J. Rosiński, "Series representations of Lévy processes from the perspective of point processes," *In Lévy Processes - Theory and Applications.* Eds. O.E. Barndorff-Nielsen, T.Mikosch and S.I. Resnick, Birkhäuser, Boston, pp. 401-415, 2001.

[41] J. Rosiński, "Tempered stable processes," *Mini-proceedings: 2nd MaPhySto Conference on Lévy Processes: Theory and Applications,* MaPhySto Miscellanea no, vol. 22, pp. 215-220, 2001.

[42] J. Rosiński, "Tempering stable processes," *Preprint*, 2004.

[43] K. Sato, "Class L of multivariate distributions and its subclasses," *J. Multivar. Anal.*, vol. 10, pp. 207-232, 1980.

[44] K. Sato, *Lévy processes and infinitely divisible distributions,* Cambridge University Press, 1999.

[45] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes,* Chapman & Hall, New York, 1994.

[46] T. Sottinen, "Fractional Brownian motion in finance and queueing," *Ph.D. dissertation*, Department of Mathematics, Faculty of Science, University of Helsinki, 2003.

[47] R. Weron, "Performance of the estimators of stable law parameters," *Research Report*, HSC/95/1, Wroclaw University of Technology, 1995

[48] W. Willinger, M. S. Taqqu, and V. Teverovsky, "Stock market prices and long-range dependence," *Finance Stochast.*, vol. 3, pp. 1-13, 1999.

Vita

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After his one-year study at the University of California at Los Angeles, he moved to Atlanta, Georgia to enroll in the School of Industrial and Systems Engineering at the Georgia Institute of Technology. In the fall of 2002, he started his doctoral research in the field of infinitely divisible distributions and Lévy processes under the direction of Prof. Christian Houdré at the School of Mathematics. In the spring of 2003, he earned two master's degrees at the Georgia Institute of Technology, M.S. in Operations Research and M.S. in Applied Mathematics. He successfully defended his doctoral dissertation, "Contributions to Infinite Divisibility for Financial Modeling", in November, 2004.